THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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VOL. XI. OCTOBER, 1922. No. 160.

GEORGE BALLARD MATHEWS, F.R.S.

G. B. MATHEWS was born in London in 1861, the son of a solicitor who later resided at Leominster in Herefordshire. He was educated at Ludlow School, at University College, London, and at St. John's College, Cambridge. He was equally distinguished at school in Classics and Mathematics, and was offered, on entering Cambridge, the Senior Scholarship either in Classics or in Mathematics, but in his second year he decided to read for the Mathematical Tripos. His private tutor was Mr. Besant, and he was Senior Wrangler in 1883. He had studied for a year at University College, of which he was afterwards a Fellow, and always spoke most gratefully of his obligations to the teaching of Professor Henrici.

My first meeting with him was on the 4th of June, 1884, when the staff of the University College of North Wales was appointed. We were chosen for the Chairs of Mathematics and Physics (he for Mathematics), and were at once thrown together in an intimate way. It soon appeared that Mathews' personality was a very remarkable and attractive one. He was a classical scholar and deeply interested in philosophical questions of all kinds, but his interest in experimental science was also keen. His mind was quick and his tongue sharp, and often his repartees in discussion were appropriate and caustic in a high degree. I remember his crushing reply to a person not remarkable for quickness of perception who raised a discussion on mass and weight, and advanced the statement that *mass* and *density* meant the same thing. "Not at all," said Mathews, "there is all the difference in the world between a massive intellect and a dense one!"

We quickly became intimate. Mathews got into the habit of discussing with me his ideas on all kinds of subjects; often we talked of matters which were strange and new and lay outside our ordinary lines of thought and work. His interest in literature was exceedingly keen and his taste was truly catholic. He subscribed to the London Library and read all the new books. Walking was his only exercise, and perhaps was not indulged in sufficiently.

In Mathematics his earliest predilections were for number theory and he formed an intention of writing a treatise on the subject. His admiration for the work of H. J. S. Smith was intense, and he read and re-read all that Smith ever wrote. Finally he produced a first volume on *The Theory of Numbers*, which was published by Deighton & Bell at his own expense. It dealt in the main with the Theory of Congruences, in which he brought the work of Gauss on this subject down to date. We looked for a continuation of the book, but none ever appeared; and indeed the range of work, for which it

would have formed about the time of its appearance an appropriate introduction, had become so transformed by new papers that the introduction itself almost required transformation. Another cause of the non-continuation of his book was the fact that his mind turned to the composition of a treatise on functions of a complex variable. At that time the excellent books on the subject now available for students did not exist, and the needs of his students weighed on his mind. But treatises began to appear and his intention of writing one was never realized. The theory of functions has made much progress in the last thirty-six years, and the need of a really good text-book was then much felt.

It perhaps ought to be stated here that I have no idea of trying to estimate the value of Mathews' mathematical work. That work I have asked should be appraised by some more competent authority. But as a collaborator with Mathews, who was associated with him in many things and in many ways, I have undertaken only the melancholy duty of writing such a personal account of him as I could, leaving the professional aspect of his life-work to be dealt with by other hands.

He had a habit of taking up different subjects from time to time, burying himself in them, as it were, and reading all the literature he could find. Thus at one time he would be absorbed in the works of Archimedes. He was specially interested in the geometrical writings which he read in the original Greek, though the hydrostatic and mensurational work of the sage to me seems the more wonderful. He tackled the *Divina Commedia* of Dante in the same way, and he had read the whole of that great poem probably several times; and he would discourse with enthusiasm, which I could admire but hardly share, of the topography of *Il Paradiso*. I do not know that he gave any attention to the other works of Dante, but he certainly made a very complete study of the *Commedia*.

His knowledge of modern languages comprised, besides Italian, French and German, of course, and he had read the great romance of Cervantes, but his interest in Spain and Spanish literature arose rather through the domination of the Moors and its connection with Arabic in the Peninsula.

In the last thirty-five years of his life he gave much attention to Arabic, and was an enthusiastic student of Arabic poetry and the great Arabic grammarians. This study originated in a challenge to myself to learn Arabic. He already read Hebrew, and was thoroughly versed in the work of Wellhausen and others on the Old Testament, in which he took a keen interest. The possession of Arabic, however, seemed to promise a wider field of interest as regards the history of early Eastern peoples. Well, we equipped ourselves with Arabic grammars and vocabularies and set to work. But, so far as I was concerned, the impulse was not sufficient to keep me going, and there were too many distractions in physics and physical mathematics. So I fell out of this Arabic amusement or work, whatever we might call it; Mathews kept on by himself, and did, I believe, some rather remarkable work in translations of some of the Arabic poets which, but for the war, might have been published. He acquired a collection of Arabic books, which he left, with his mathematical library, to the University College of North Wales.

In 1896 Mathews became profoundly discouraged by the attitude of Welsh students to Mathematics, and of the students of the new Day Training Department of Teachers to himself. There was nothing that need have been taken very seriously; it was, I think, merely one of those transitory waves of feeling which arise everywhere in university life with regard to particular teachers or particular subjects, which are unworthy of the serious attention of any capable and devoted teacher, and Mathews was emphatically a man who spent himself in every possible way for his students. Some of the students had become possessed of the silly and conceited notion that the study of philosophy, especially in the Neo-Hegelian system, was a mark of genius,

while to be mathematical was to have a base mechanical turn of mind, and the absence of mathematical ability showed that their minds were "speculative" and therefore of a higher cast, and fitted them better for great careers, just as a "First in Greats" at Oxford, with its philosophical implications, was supposed to be the first step in the career of a proconsul. It is certain that, as Mathews used to remark to me, people are not ashamed of, but are rather proud of, any mathematical inability, and he wrote an article, which was published posthumously in *Nature*, on this psychological peculiarity of many groups of people (ungrateful as they certainly were in this case).

I combated as best I could his impressions and resolutions. Such fashions and feelings pass, and the views and still less those thoughtless and cruel acts, which mark the sheeplike instinct of the members of a crowd, are, if not to be absolutely ignored, not worthy to be seriously noticed, and ought not to be allowed to produce profound mental disquietude. The disappointment was too great, and he insisted on resigning, and left Bangor for a time to settle at Cambridge, where he was shortly after appointed to a University Lectureship in Mathematics. Here again his disappointment was revived. Only a very small number of men attended the lectures which, with the approval of the authorities, he undertook to give. It was not expected that the attendance on special subjects should be more than a very few: it never is. Again he retired, and finally settled down in a special lectureship in his old department at Bangor.

His life was a rather solitary one. He smoked and worked and read in his rooms when he would have been better in the open air; but he was not a mere recluse. He much enjoyed talking over current events with his friends. I remember that when he stayed with me in Strathspey in 1913 he was eloquent in appreciation of the character of Lord Kitchener, and talked with much vigour and information of all Kitchener's work for Egypt.

Mathews was exceedingly sensitive, and almost morbidly afraid of appearing to put himself forward in any way, so that he hardly received the recognition which was his due. He was elected to the Royal Society in 1897, and later served on the Council. He was also President of the London Mathematical Society, but with a fair amount of the "push" and "go" of the time, which he detested, he might have bulked much more largely in the public eye. But he was a great scholar, a great mathematician, and a great gentleman. In pure algebra there was no greater authority, and his numerous articles and reviews in *Nature*, all signed "G. B. M.," are very valuable to students and to all mathematicians. The idea of republishing these articles has been mooted, and it is to be hoped that it may sooner or later be carried out. He wrote graceful essays after the style of Charles Lamb, and one or two of these which appeared in the *College Magazine* showed much taste, and, like all he wrote, a great command of vigorous nervous English. He received the honorary degree of LL.D. from the University of Glasgow in 1915.

At the end of the war years Mathews' health became unsatisfactory, a result he himself thought of the bad quality of the food of those years. He underwent in 1919 a very serious operation, and his friends thereafter were gravely concerned regarding him. He recovered partially from the disease, and hopes were entertained that a certain course of treatment would bring about a permanent cure. On March 19 he sustained a seizure, and died in the Nursing Home in Liverpool, in which he had spent the closing months of 1921 and the beginning of this year.

I have now, and very imperfectly, carried out my task, and have given, I hope, an account of perhaps the most intimate friend I ever had to which he himself would not have taken grave exception. A. GRAY.

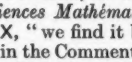
THE ST. ANDREW'S CROSS (X) AS A MATHEMATICAL SYMBOL.

By Professor F. Cajori, University of California.

In Mathematics crosses are not signs of illiteracy, but symbols of varied productive achievement.

It is well known that the St. Andrew's cross (X) occurs as the symbol for multiplication in W. Oughtred's *Clavis Mathematicae*, 1631, and also (in the form of the letter X) in an anonymous Appendix which appeared in E. Wright's 1618 edition of John Napier's *Descriptio*. This Appendix is very probably from the pen of Oughtred. The question has arisen, is this the earliest use of X to designate multiplication? It has been answered in the negative—incorrectly so, we think, as we shall endeavour to show.

In the *Encyclopédie des Sciences Mathématiques*, T. I., vol. 1, 1904, p. 40, note 158, we read concerning X, "we find it between the factors of a product, placed one beneath the other, in the Commentary added by Oswald Schreckenfuchs to Ptolemy's *Almagest*, 1551." [1] As will be shown more fully later, this is not a correct interpretation of the symbolism. Not two, but four numbers are involved, two in a line and two others immediately beneath, thus:



The cross does not indicate the product of any two of these numbers, but each bar of the cross connects two numbers which are multiplied. One bar indicates the product of 315172 and 174715, the other bar the product of 395093 and 295448. Each bar is used as a symbol singly, the two bars are not considered here as one symbol.

Another reference to the use of X before the time of Oughtred is made by E. Zirkel [2] of Heidelberg, in a brief note in which he protests against attributing the "invention" of X to Oughtred; he states that it had a period of development of over 100 years. Zirkel does not make his position clear, but if he does not mean that X was used before Oughtred as a sign of multiplication, his protest is pointless.

Our own studies have failed to bring to light a clear and conclusive case where, before Oughtred, X was used as a symbol of multiplication. In medieval manuscripts and early printed books X was used as a mathematical sign, or a combination of signs, in ten or more different ways, as follows: (1) In solutions of problems by the process of two false positions, (2) In solving problems in compound proportion involving integers, (3) In solving problems in simple proportion involving fractions, (4) In the addition and subtraction of fractions, (5) In the division of fractions, (6) In checking results of computation by the processes of casting out the 9's, 7's, or 11's, (7) As part of a group of lines drawn as guides in the multiplication of one integer by another, (8) In computing on lines, to mark the line indicating "thousands," (9) To take the place of the multiplication table above 5 times 5, (10) In dealing with amicable numbers. We shall briefly discuss each of these in order.

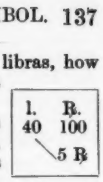
(1) *The Process of Two False Positions.* The use of X in this process is found in the *Liber abaci* of Leonardo [3] of Pisa, written in 1202. We must begin by explaining Leonardo's use of a single line or bar. A line connecting two numbers indicates that the two numbers are to be multiplied together.

[1] *Clavdii Ptolemaei Pelusiensis Alexandrini Omnia quae extant Opera.* Basileae, 1551, Lib. ii. Annotationes.

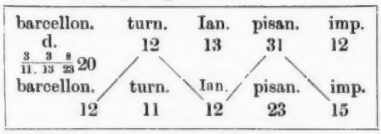
[2] Emil Zirkel, *Zeitschr. f. math. u. naturw. Unterricht*, vol. 52, 1921, p. 96. An article on the sign ×, which we have not seen and which we know only from references to it made by other writers, is N. L. W. A. Gravelaar in *Wiskundig Tijdschrift*, i. (1904), pp. 122-124.

[3] Leonardo of Pisa, *Liber Abaci*, 1202. Edited by B. Boncompagni, Roma, 1857, vol. 1, p. 84.

In one place[1] he solves the problem, if 100 rotuli are worth 40 libras, how many libras are 5 rotuli worth? On the margin of the sheet stands the following.



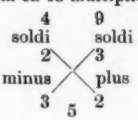
The line connecting 40 and 5 indicates that the two numbers are to be multiplied together. Their product is divided by 100, but no symbolism is used to indicate the division. Leonardo uses single lines over a hundred times in the manner here indicated. In more complicated problems he uses two or more lines, but they do not necessarily form crosses. In a problem involving five different denominations of money he gives the following diagram.[1]



Here the answer 20 + is obtained by taking the product of the connected numbers and dividing it by the product of the unconnected numbers.

Leonardo uses a cross in solving, by double false position,[2] the problem, if 100 rotuli cost 13 libras, find the cost of 1 rotulus. The answer is given in solidi and denarii, where 1 libra = 20 solidi, 1 solidus = 12 denarii. Leonardo assumes at random the tentative answers (the two false positions) of 3 solidi and 2 solidi. But 3 solidi would make the cost of 100 rotuli 15 libra, an error of +2 libras; 2 solidi would make this cost 10, an error of −3. By the underlying theory of two false positions, the errors in the answers (*i.e.* the errors $x-3$ and $x-2$ solidi) are proportional to the errors in the cost of 100 rotuli (*i.e.* +2 and −3 libras); this proportion yields $x=2$ solidi and $7\frac{1}{5}$ denarii. If the reader will follow out the numerical operations for determining our x, he will understand the following arrangement of the work given by Leonardo:

"Additum ex 13 multiplicationibus



Additum ex erroribus."

Observe that Leonardo very skilfully obtains the answer by multiplying each pair of numbers connected by lines, thereby obtaining the products 4 and 9, which are added in this case, and then dividing 13 by 5 (the sum of the errors). The cross occurring here is not *one* symbol, but *two* symbols. Each line singly indicates a multiplication. It would be a mistake to conclude that the cross is used here as a symbol expressing multiplication.

The use of two lines crossing each other, in double or single false position, is found in many authors of later centuries. For example, it is used by the Germans Johann Widman[3] (1489) and Ch. Clavius[4] (1611), the Italian Luca Pacioli[5] (1494), the Englishmen Cuthbert Tonstall[6] (1522) and Robert

[1] *Op. cit.* vol. 1, p. 127.

[2] *Op. cit.* vol. 1, p. 319.

[3] Johann Widman, *Behēde vnd hubsche Rechenung*, Leipzig, 1489. We have used J. W. L. Glaisher's article in *Messenger of Mathematics*, vol. 51, 1921, p. 16.

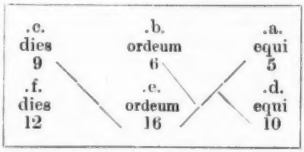
[4] Christophori Clavii Bambergensis, *Opera Mathematica*, Mogvntiae, 1612. Tomus secundus, Numeratio, p. 58.

[5] L. Pacioli, *Summa de arithmetica, geometria*, etc., 1494. We have used the 1523 edition, printed at Toscolano, folio 99[b], 100[a], 182.

[6] C. Tonstall, *De arte supputandi*, 1522. We have used the Strassburg edition of 1544, p. 393.

Recorde [1] (1543 ?), the German Spenlin [2] (1546), the Spaniard Hortega [3] (1552), the Frenchman Jan Trenchant [4] (1578). That this use continued through later periods of time and in widely separated parts of the world is evident, if we quote the British writers S. Jeake [5] (1696) and E. Wingate [6] (1704), the Italian Guido Grandi [7] (1740), the Americans D. Adams [8] and J. Preston.[9]

(2) *Compound Proportion with Integers.* We begin again with Leonardo of Pisa [10] (1202) who gives the problem: If 5 horses eat 6 quarts of barley in 9 days, for how many days will 16 quarts feed 10 horses ? His numbers are arranged thus :



The answer is obtained by dividing $9 \times 16 \times 5$ by the product of the remaining known numbers. Answer 12. Of the four lines drawn the last three form a cross. But the cross itself is not used as a symbol; its formation is here accidental.

Somewhat different applications of lines crossing each other are given by Nicolas Chuquet [11] (1484) and Luca Pacioli [12] in dealing with numbers in continued proportion.

Chuquet finds two mean proportionals between 8 and 27 by the scheme

where 12 and 18 are the two mean proportionals sought; that is, 8, 12, 18, 27 are in continued proportion.

(3) *Proportions involving Fractions.* Lines forming a cross (X), together with two horizontal parallel lines, were extensively applied to the solution of proportions involving fractions, and constituted a most clever device for obtaining the required answer mechanically. If it is the purpose of mathematics to resolve complicated problems by a minimum mental effort, then this device takes high rank.

The very earliest arithmetic ever printed, namely the anonymous booklet brought out in 1478 at Treviso,[13] in northern Italy, contains an interesting

[1] R. Recorde, *Grovnd of Artes*, 1543 (?). We have used an edition issued between 1636 and 1646 (title page missing).

[2] Gall Spenlin, *Arithmetica künstlicher Rechnung*, 1546. We have used J. W. L. Glaisher's article in *Messenger of Mathematics*, vol. 51, 1921, p. 62.

[3] Juan de Hortega, *Tractado subtilissimo de arismetica y de geometria*, emendado por Lonçalo Busto, 1552, folio 138, 215b.

[4] *L'arithmétique de Jan Trenchant*, 4 éd., Lyon, 1578, p. 216.

[5] Samuel Jeake, ΛΟΓΙΣΤΙΚΗΛΟΓΙΑ *or Arithmetick*, London, 1696 (Preface 1674), p. 501.

[6] *Mr. Wingate's Arithmetick*, enlarged by John Kersey, 11th ed., with supplement by George Shelley, London, 1704, p. 128.

[7] Guido Grandi, *Instituzioni di arithmetica pratica*, Firenze, 1740, p. 104.

[8] Daniel Adams, *Scholar's Arithmetic*, 10th ed., Keene, N.H., 1816, p. 199.

[9] John Preston, *Lancaster's Theory of Education*, Albany [New York], 1817, p. 349.

[10] Leonardo of Pisa, *loc. cit.* p. 132.

[11] Nicolas Chuquet, *Le Triparty en la Science des Nombres*, edited by A. Marre, in *Bullettino Boncompagni*, vol. 13, 1880; reprint, Roma, 1881, p. (115).

[12] Luca Pacioli, *op. cit.* folio 93a.

[13] The Treviso arithmetic of 1478 is described and partly given in facsimile by B. Boncompagni in *Atti dell' Accademia Pontificia dei nuovi Lincei*, tome 16, 1862-63, Roma 1863. See p. 563.

problem of two couriers starting from Rome and Venice, respectively, the Roman reaching Venice in 7 days, the Venetian arriving at Rome in 9 days. If Rome and Venice are 250 miles apart, in how many days did they meet, and how far did each travel before they met? They meet in $3\frac{1}{16}$ days. The computation of the distance travelled by the courier from Rome, calls for the solution of the proportion which we write $7 : 250 = \frac{63}{16} : x$.

The Treviso arithmetic gives the following arrangement:

$$\begin{array}{c} 112 \\ 7 \quad\quad 250 \text{——} 63 \\ 1 \quad\quad\quad 1 \text{ ——} 16 \end{array}$$

The connecting lines indicate what numbers shall be multiplied together, namely 1, 250 and 63, also 7, 1 and 16. The product of the latter, namely 112, is written above, on the left. The author then finds $250 \times 63 = 15750$ and divides this by 112, obtaining $140\frac{5}{8}$ miles.

These guiding lines served as Ariadne threads through the maze of a proportion involving fractions.

We proceed to show that this magical device was used again by Chuquet (1484), Widman (1489) and Pacioli (1494). Thus Chuquet [1] uses the cross in the problem to find two numbers in the ratio of $\frac{3}{4}$ to $\frac{2}{3}$ and whose sum is 100. He writes $\begin{array}{cc} 3 & 2 \\ \frac{}{4} & \frac{}{3} \end{array}$; multiplying 3 by 3, and 2 by 4, he obtains two numbers in the proper ratio. As their sum is only 17, he multiplies each by $\frac{100}{17}$ and obtains $47\frac{1}{17}$ and $52\frac{16}{17}$.

Johann Widman [2] solves the proportion $9 : \frac{53}{8} = \frac{89}{8} : x$ in this manner: "Secz also $\begin{array}{cc} 9 & 53 \text{——} 89 \\ \frac{}{1} & \frac{}{8} \text{——} \frac{}{8} \end{array}$ machss nach der Regel vnd küpt 8 fl. 35s 9 helr $\frac{5}{12}$." It will be observed that the computor simply took the products of the numbers connected by lines. Thus $1 \times 53 \times 89 = 4717$ gives the numerator of the fourth term; $9 \times 8 \times 8 = 576$ gives the denominator. The answer is 8 florins and a fraction.

Such settings of numbers are found in Luca Pacioli [3] (1494), Ch. Rudolph [4] (1526), O. Schreckenfuchs,[5] Hortega [6] (1552), Tartaglia [7] (1556), J. Trenchant [8] (1578), J. Perez de Moya [9] (1784). It is remarkable that in England neither Tonstall nor Recorde used this device. Recorde [10] and Leonard Digges [11] use a slightly different and less suggestive scheme, namely the capital letter Z for proportions involving either integers or fractions. Thus, $3 : 8 = 16 : x$ is given by Recorde in the form $\begin{array}{cc} 3 & \text{——} 16 \\ 8 & \diagup \end{array}$.

(4) *Addition and Subtraction of Fractions.* Perhaps even more popular than in the solution of proportions involving fractions was the use of guiding

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[1] Chuquet, *loc. cit.* p. (84).

[2] Johann Widman, *op. cit.*; see J. W. L. Glaisher, *loc. cit.* p. 6.

[3] Luca Pacioli, *loc. cit.*, 1523, folio 18, 27, 54, 58, 59, 64.

[4] Christoph Rudolph, *Kunstliche Rechnung*, 1526. We have used one of the Augsburg editions, 1574 or 1588 (title-page missing), GV11.

[5] O. Schreckenfuchs, *op. cit.* 1551. [6] Juan de Hortega, *op. cit.* 1552, folio 92a.

[7] Tartaglia, *General Trattato di Nvmeri*, la prima parte, 1556, folio 111b, 117a.

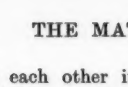
[8] J. Trenchant, *op. cit.* p. 142.

[9] Juan Perez de Moya, *Arithmetica*, Madrid, 1784, p. 141. This text is the same as that of the edition that appeared in Salamanca in 1562.

[10] Robert Recorde, *op. cit.* p. 175.

[11] [Leonard Digges], *A Geometrical Practical Treatize named Pantometria*, London, 1591.

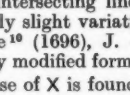
lines crossing each other in the addition and subtraction of fractions. Chuquet[1] in 1484 represents the addition of $\frac{2}{3}$ and $\frac{4}{5}$ by the following scheme:



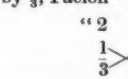
The lower horizontal line gives $3 \times 5 = 15$; we have also $2 \times 5 = 10$, $3 \times 4 = 12$; hence the sum $\frac{22}{15} = 1\frac{7}{15}$.

The same line-process is found in L. Pacioli[2] (1494), Ch. Rudolph[3] (1526), Apianus[4] (1527), F. Feliciano[5] (1550), O. Schreckenfuchs[6] (1551), Hortega[7] (1552), Chr. Clavius[8] (1611). In England Tonstall and Recorde do not employ this intersecting line-system, but Edmund Wingate[9] avails himself of it, with only slight variations in the mode of using it. We find it also in Samuel Jeake[10] (1696), J. P. de Moya[11] (1784), J. B. Corachan[12] (1719), and in slightly modified form in G. E. Crusoe[13] (1921).

A similar use of X is found in reducing two radicals, say $\sqrt[3]{4x}$ and $\sqrt{8x}$, to radicals of the same order, $\sqrt[6]{16x^2}$ and $\sqrt[6]{512x^3}$. Peletier[14] in 1560 represents the process thus:

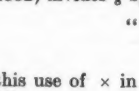


(5) *Division of Fractions.* Less frequent than in the preceding processes is the use of lines in the multiplication or division of fractions, which called for only one of the two steps taken in solving a proportion involving fractions. In dividing $\frac{1}{3}$ by $\frac{1}{2}$, Pacioli[15] (1494) writes



and obtains $\frac{3}{2}$ or $1\frac{1}{2}$.

Petrus Apianus (1527) uses parallel horizontal lines to indicate the processes in multiplication of fractions, thus, $\frac{1}{2}$——$\frac{3}{5}$; he uses X in division. Juan de Hortega[16] (1552) divides $\frac{3}{8}$ by $\frac{5}{6}$, according to the following scheme:



We find this use of × in Samuel Jeake,[17] in Blundevile[18] and J. Perez de Moya[19] (1784).

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[1] Nicolas Chuquet, *loc. cit.* p. (54).
[2] Luca Pacioli, *loc. cit.* 1523, folio 51, 52, 53.
[3] Christoph Rudolph, *op. cit.*, under addition and subtraction of fractions.
[4] Petrus Apianus, *Kauffmanes Rechnung*, Ingolstadt, 1527.
[5] Francesco Feliciano, *Libro di Arithmetica e geometria*, 1550.
[6] O. Schreckenfuchs, *op. cit.*, Annot., folio 25b.
[7] Hortega, *op. cit.*, 1552, folio 46, 55a, 63b.
[8] Chr. Clavius, *Opera Omnia*, Tom 1, 1611 Euclid, p. 383.
[9] E. Wingate, *op. cit.* 1704, p. 152.
[10] Samuel Jeake, *op. cit.* p. 51.
[11] J. P. de Moya, *op. cit.* p. 103.
[12] Juan Bautista Corachan, *Arithmetica demonstrada*, Barcelona, 1719, p. 87.
[13] George E. Crusoe, *Y Mathematics* (Why Mathematics ?), Pittsburgh, Pa., 1921, p. 21.
[14] *Jacobi Peletarii Cenomani, de occvlta parte nvmervorvm, qvam Algebram vocant, Libri duo.* Parisiis, 1560, folio 52 b.
[15] L. Pacioli, *op. cit.* 1523, folio 54a.
[16] Juan de Hortega, *op. cit.* 1552, folio 66a.
[17] Samuel Jeake, *op. cit.* p. 58.
[18] *Mr. Blundevil. His Exercises contayning eight Treatises*, London, 1636, p. 20.
[19] Juan Perez de Moya, *op. cit.* p. 117.

(6) *Casting Out the* 9's, 7's, *or* 11's. Checking results by casting out the 9's was far more common in old arithmetics than by casting out the 7's or 11's. Two intersecting lines afforded a convenient grouping of the four results of an operation. Sometimes the lines appear in the form X, at other times in the form +. Luca Pacioli [1] divides 97535399 by 9876, and obtains the quotient 9876 and remainder 23. Casting out the 7's (*i.e.* dividing a number by 7 and noting the residue) he obtains for 9876 the residue 6, for 97535399 the residue 3, for 23 the residue 2. He arranges these residues thus: " $\frac{6}{6}\,\big|\,\frac{2}{3}$ "

Observe that multiplying the residues of the divisor and quotient, 6 times 6 = 36, one obtains 1 as the residue of 36. Moreover 3 − 2 is also 1. This completes the check.

Nicolas Tartaglia [2] checks, by casting out the 7's, the division 912345 ÷ 1987 = 459 and remainder 312.

Casting the 7's out of 912345 gives 0, out of 1987 gives 6, out of 459 gives 4, out of 312 gives 4. Tartaglia writes down " $\frac{4}{6}\,\big|\,\frac{4}{0}$ "

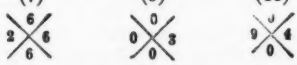
Here 4 times 6 = 24 yields the residue 3; 0 minus 4, or better 7 minus 4, yields 3 also. The result " checks."

Would it be reasonable to infer that two perpendicular lines + signified multiplication? We answer No, for, in the first place, the authors do not state that they attached this meaning to the symbols, and, in the second place, such a specialized interpretation does not apply to the other two residues in each example, which are to be *subtracted* one from the other. The more general interpretation, that the lines are used merely for the convenient grouping of the four residues, fits the case exactly.

Ch. Rudolph [3] checks the multiplication 5678 times 65 = 369070 by casting out the 9's (*i.e.* dividing the sum of the digits by 9 and noting the residue); he finds the residue for the product to be 7, for the factors to be 2 and 8. He writes down " "

Here 8 times 2 = 16, yielding the residue 7, written above. This residue is the same as the residue of the product; hence the check is complete. It has been argued that in cases like this Rudolph used X to indicate multiplication. This interpretation does not apply to other cases found in Rudolph's book (like the one which follows) and is wholly indefensible. We have previously seen that Rudolph used X in the addition and subtraction of fractions. Rudolph checks the proportion $9 : 11 = 48 : x$, where $x = 58\frac{2}{9}$, by casting out the 7's, 9's and 11's as follows:

" (7) (9) (11) "



Take the check by 11's (*i.e.* division of a number by 11 and noting the residue). It is to be established that $9x = 48$ times 11, or that 9 times 528 = 48 times 99. Begin by casting out the 11's of the factors 9 and 48; write down the residues 9 and 4. But the residues of 528 and 99 are both 0. Multiplying the residues 9 and 0, 4 and 0, we obtain in each case the product 0. This is shown in the figure. Note that here we do not take the product 9 times 4; hence X could not possibly indicate 9 times 4.

The use of X in casting out the 9's is found also in Recorde's *Grovnd of Artes* and in Clavius [4] who casts out the 9's and also the 7's.

[1] Luca Pacioli, *op. cit.* 1523, folio 35. [2] N. Tartaglia, *op. cit.* 1556, folio 346.
[3] Chr. Rudolph, *Kunstliche Rechnung*, Augsburg, 1574 or 1588 edition.
[4] Chr. Clavius, *Opera Omnia*, 1612, tom. 1, 1611, Numeratio, p. 11.

Hortega [1] (1552) follows the Italian practice of using the lines +, instead of X, for the assignment of resting places for the four residues considered. Nicolas Hunt [2] (1633) uses the Latin cross ⊣+. The regular X is used by Th. York [3] (1687), John Ayres [4] (1730) and Benjamin Workman [5] and others.

In the more recent centuries the use of a cross in the process of casting out the 9's has been abandoned almost universally; we have found it given however, in an English mathematical dictionary [6] of 1814 and in a twentieth century Portuguese cyclopedia.[7]

(7) *Multiplication of Integers.* In Pacioli (1494) the square of 37 is found mentally with the aid of lines indicating the digits to be multiplied together, thus:

1369

From the lower 7 two lines radiate, indicating 7 times 7, and 7 times 3. Similarly for the lower 3. We have here a cross as part of the line-complex. In squaring 456 a similar scheme is followed; from each digit there radiate in this case three lines. The line-complex involves three vertical lines and three well-formed crosses X.

There are cases on record where the vertical lines are omitted, either as deemed superfluous or as the result of an imperfection in the type-setting. Thus an Italian writer, Joseppo Vnicorno [8] writes:

" 7 8 "
X
5 6
4368

It would be a rash procedure to claim that we have here a use of X to indicate the product of two numbers; these *lines* indicate the product of 6 and 70, and of 50 and 8; the lines are not to be taken as *one* symbol; they do not mean 78 times 56.

A procedure resembling that of Pacioli, but with the lines marked as arrows, is found in a recent text by G. E. Crusoe.[9]

(8) *To Mark the Place for "Thousands."* In old arithmetics explaining the computation upon lines (a modified abacus mode of computation), the line on which a dot signified "one thousand" was marked with a X. The plan is as follows:

X ———1000
500
———100
50
———10
5
———1

This notation was widely used in Continental and English texts.

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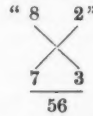
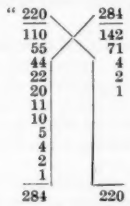
[1] Juan de Hortega, *op. cit.* folio 42b.
[2] Nicolas Hunt, *Hand-Maid to Arithmetick*, London, 1633.
[3] Tho. York, *Practical Treatise of Arithmetick*, London, 1687, p. 38.
[4] John Ayres, *Arithmetick made Easie*, by E. Hatton, London, 1730, p. 53.
[5] Benjamin Workman, *American Accountant*, Philadelphia, 1789, p. 25.
[6] Peter Barlow, *Math. & Phil. Dictionary*, London, 1814, art. "Multiplication."
[7] *Encyclopedia Portugueza*, Porto, 19— ?, vol. 7, art. "Nove."
[8] S. Joseppo Vnicorno, *De l'arithmetica universale*. In Venetia, 1598, fo. 20. Quoted from C. Le Paige, "Sur l'origine de certains Signes d'Opération," *Annales de la société scientifique de Bruxelles*, 16th year, 1891-92, Part II., p. 82.
[9] George E. Crusoe, *op. cit.* p. 6.

(9) *In Place of Multiplication Table above* 5×5. This old procedure is graphically given in Recorde's *Grovnd of Artes* (1543 ?). Required to multiply 7 by 8. Write the 7 and 8 at the cross as shown here; next, $10 - 8 = 2$, $10 - 7 = 3$; write the 2 and 3 as shown:



Then, $2 \times 3 = 6$, write the 6. $7 - 2 = 5$, write the 5. The required product is 56. We find this process again in Hudalrich Regius[1] (1543), Th. York[2] (1687), and Samuel Jeake[3] (1696) and many others.

(10) *Amicable Numbers.* N. Chuquet[4] shows graphically that 220 and 284 are amicable numbers (each the sum of the factors of the other) thus:



The old graphic aids to computation which we have described are interesting as indicating the emphasis that was placed by early arithmeticians upon devices that appealed to the eye and thereby contributed to economy of mental effort.

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GLEANINGS FAR AND NEAR.

138. Coordinates. We do not find the word in Schooten, Beaune, or others of the immediate school of Descartes. De Witt calls the abscissa *crus patiens*, and the ordinate *crus efficiens*. Coordinates (so called) are used in the writings of John Bernoulli, but in Newton the phrase for them is 'lineae ordinatim applicatae."—De Morgan, [*Co-ordinates, P.C.* 1837.]

139. Thomas Rymer gives Desdemona as "a lesson to Husbands, that, before their Jealousie be tragical, the proofs may be mathematical."—? In Preface to translation of Rapin's *Reflections on Aristotle's 'Treatise of Poesie.'*

140. (Porson) was not, like Napoleon, a great proficient in mathematics; but he had, like him, a memory as retentive of numbers as of everything else. He occasionally took an unaccountable pleasure in repeating large sums, and had a tiresome trick of counting them aloud, especially after he had drunk much wine, and was disposed by such a test to prove to himself that his faculties were not sensibly impaired thereby.—Lord Holland's *Further Memories of the Whig Party*, 1807-1821 (1905, pp. 331-2).

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[1] Hudalrich Regius, *Vtriusqve Arithmetices Epitome*, Fribvrgi Prisgoiae, 1543, folio 61.

[2] Tho. York, *op. cit.* p. 35.

[3] Samuel Jeake, *op. cit.* p. 22.

[4] N. Chuquet, *op. cit.* p. (69).

IDENTITIES IN ELEMENTARY TRIGONOMETRY.

By F. G. Hall, M.A.

If it be granted that the study of trigonometry in the secondary school course should not be limited to the numerical solution of triangles but should include some treatment of *form* as an introduction to higher mathematical analysis, then the importance of the trigonometrical identity must be admitted. A certain amount of valuable training can be obtained from the proper study of the identity if this study be systematic; but there seems little to be gained by working through the customary set of miscellaneous examples inserted at odd intervals in the trigonometrical text-book.

This article aims at a rough classification of the more important types, and at indicating how they may best be approached. It is, I feel sure, of more educational value for a boy to "discover" possible simplifications and transformations than for him to "prove" what often seem rather useless and very cumbersome results.

(1) As soon as a boy is familiar with the definitions of the six ratios—and so far as this article is concerned it is immaterial whether he has begun with the study of one particular ratio first or with all six together—he should be shown that as he knows a relation connecting the squares on the sides of a right-angled triangle he may expect relations amongst the six trigonometrical ratios. He may thus be led to deduce from the relation $a^2+b^2=c^2$ the important results $a^2/c^2+b^2/c^2=1$, etc., etc.

The three important identities thus obtained, and the other simpler results following directly from the definitions of the ratios, he must learn to use in simplifying or transforming trigonometrical expressions. Practice at this stage might well be restricted to illustrations of how simplification or transformation may easily be effected. Suitable types of questions are:

(*a*) Simplify $\dfrac{\cos A+\cos B}{\sin A-\sin B}+\dfrac{\sin A+\sin B}{\cos A-\cos B}$; $\dfrac{\tan A+\tan B}{\cot A+\cot B}$;

$$\sin^2 A\,.\tan A-\cos^2 A\,.\cot A;\quad \frac{\sin\theta+2\sin\theta\,.\cos\theta}{1+\cos\theta+\cos^2\theta-\sin^2\theta}.$$

(*b*) Express $(\sec A\pm\tan A)^2$ in terms of $\sin A$;

$$\frac{5\sin\alpha+7\cos\alpha}{6\cos\alpha-3\sin\alpha} \quad ,, \quad ,, \quad \tan\alpha.$$

Show that $\cos^4 A-\sin^4 A+2\sin^2 A$ has always the same value whatever may be the value of A.

(2) The next stage at which the study of trigonometrical form should be resumed is when the subject of the ratios of compound angles is reached. In the meantime much work, of course, of a more practical nature will have been done in trigonometry.

To find the value of $\sin(A+B)$ in terms of the ratios of A and B, I prefer the following method: Starting with a ray OP of length r inclined at an angle $(A+B)$ to some fixed line, a line representing $r\,.\sin(A+B)$ is easily found. Other lines representing $r\,.\cos B\,.\sin A$ and $r\,.\sin B\,.\cos A$ are then discovered, and it is easily seen that the first line is equal to the sum of the other two; whence the identity follows at once.

The boy may then be asked to find a corresponding expansion for $\cos(A+B)$, and to see what the two formulae become when $-B$ is substituted for B.

We are not here concerned with the practical use of the addition formulae, nor with the important results obtained by putting $B=A$ or $B=2A$. Some

time may well be spent in showing how, by their use, an expression of the form $a\cos\theta + b\sin\theta$ can be transformed into $r.\sin(\theta+\alpha)$ or how

$$\frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta}$$

is transformed into $\tan(\theta+\alpha)$; but the formulae

$$2.\sin A.\cos B = \sin(A+B) + \sin(A-B), \text{ etc.},$$

which are deduced from them, are probably of much greater value from the point of view of this article.

The pupil should be well practised in applying this last set of identities to the simplification of expressions such as $\Sigma \sin A.\sin(B-C)$—and this is a very convenient stage at which to introduce into trigonometrical work the most useful Σ notation. A careful study of expressions of the above type—including perhaps $\Sigma \sin(B+C).\sin(B-C)$ and the allied expressions

$$\Sigma \sin(B+C).\cos(B-C), \quad \Sigma \cos(B+C).\sin(B-C)$$

and

$$\Sigma \cos(B+C).\cos(B-C)$$

—will do more to lead a boy to the appreciation of trigonometrical form than the "proof" of any number of identities such as:

$$\frac{\sin(x+2y) - 2\sin(x+y) + \sin x}{\cos(x+2y) - 2\cos(x+y) + \cos x} = \tan(x+y).$$

(3) The last stage, so far as secondary school work is concerned, is reached when the results of (2) are applied to the expressions

$$4\sin A.\sin B.\sin C, \quad 4\sin A.\sin B.\cos C, \quad 4\sin A.\cos B.\cos C$$

and

$$4\cos A.\cos B.\cos C.$$

The equivalent values of these can easily be "discovered" once the method of treatment has been indicated: $4\cos A.\cos B.\cos C$, for example, is to be thought of as $2\cos A(2\cos B.\cos C)$, the bracket is to be expressed as the sum of two cosines, and hence the whole expression as the sum of four cosines.

A good pupil will appreciate the fact that the symmetrical expressions

$$4\sin A.\sin B.\sin C \quad \text{and} \quad 4\cos A.\cos B.\cos C$$

lead to the symmetrical results

$$-\sin(A+B+C) + \Sigma \sin(B+C-A) \quad \text{and} \quad \cos(A+B+C) + \Sigma \cos(B+C-A),$$

whereas $4\sin A.\sin B.\cos C$ and $4\sin A.\cos B.\cos C$

can be expressed only as

$$\cos(B+C-A) + \cos(C+A-B) - \cos(A+B-C) - \cos(A+B+C)$$

and $-\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) + \sin(A+B+C)$

respectively.

The pupil should then deduce the corresponding results for the special case in which $A+B+C=180°$:

$$4\sin A.\sin B.\sin C = \Sigma \sin 2A,$$
$$4\sin A.\sin B.\cos C = -\cos 2A - \cos 2B + \cos 2C + 1,$$
$$4\sin A.\cos B.\cos C = -\sin 2A + \sin 2B + \sin 2C,$$
$$4\cos A.\cos B.\cos C = -1 - \Sigma \cos 2A.$$

I think it is best for the pupil to meet these results first as special cases of the more general formulae. If it is desired to prove any one of them separately, the same method of starting from the product of the sines or cosines should be employed. This treatment will often succeed with much harder examples—examples too often solved by methods which the tyro finds it impossible to imitate.

Example (Hobson's *Plane Trigonometry*, p. 78, No. 9):

If $\alpha+\beta+\gamma=\pi$ and $\tan\frac{1}{2}(\beta+\gamma-\alpha).\tan\frac{1}{2}(\gamma+\alpha-\beta).\tan\frac{1}{2}(\alpha+\beta-\gamma)=1$, prove that $1+\cos\alpha+\cos\beta+\cos\gamma=0$.

From the second relation, using the above *general* formulae, we have at once:

$$\frac{-\sin\frac{1}{4}(\alpha+\beta+\gamma)+\Sigma\sin\frac{1}{4}(3\alpha-\beta-\gamma)}{\cos\frac{1}{4}(\alpha+\beta+\gamma)+\Sigma\cos\frac{1}{4}(3\alpha-\beta-\gamma)}=1.$$

If now $\alpha+\beta+\gamma=\pi$, this becomes:

$$\frac{-\frac{1}{\sqrt{2}}+\Sigma\sin\left(\alpha-\frac{\pi}{4}\right)}{\frac{1}{\sqrt{2}}+\Sigma\cos\left(\alpha-\frac{\pi}{4}\right)}=1,$$

which easily reduces to $1+\Sigma\cos\alpha=0$.

I shall end by pointing out that other identities, often set as exercises, can easily be deduced from the above general formulae or proved separately by the same method.

Substituting $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$ for A, B, C, and then introducing the relation $A+B+C=180°$, we find:

$$4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}=-1+\Sigma\cos A,$$

$$4\sin\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}=\sin A+\sin B-\sin C,$$

$$4\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}=-\cos A+\cos B+\cos C+1,$$

$$4\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}=\Sigma\sin A.$$

F. G. HALL.

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141. Mathematics—A Challenge to all England. If any man can resolve the following problem, let him send his answer (postage paid) to me, and I will inform the public. *N.B.*—I will not have an interview with any one.

Problem—To divide any given number (say $2\frac{2}{3}$) into 10 different numbers, as follows: The two first numbers are to be such, that, if you square the sum of the greater and lesser, it will equal the sum of the squares of each, with twice the product of the two first.

The next three numbers are to be such, that, if you multiply the sum of the greater and lesser numbers by the next greater number, and to this product add the square of the lesser number, it will equal the square of the greater number. The next two numbers are to be such that, if to the square of their sum, you add the square of the lesser number, it will equal the sum of the two multiplied by the lesser; to the double of which product you must add the square of the greater; also, they are to be two such numbers, that if their product be added to the sum of their squares, it shall make a square number. The three last numbers are to be such, that, if to the square of the sum of the greater number you add the square of the lesser, it will double the sum of the squares of the greater numbers; moreover, if you square the sum of the third and ninth numbers, and also add the sum of the squares of the fifth and eighth numbers, the sum of the squares of the whole shall be a square number; but if you cube the fourth, fifth, and eighth numbers, the sum of their cubes shall be a cube number—required the ten numbers.—Hone's *Y.B.*, July 31, 1832, p. 447.

G. F. A. B., 3 Northampton Street, Clerkenwell, Tuesday, Dec. 7, 1830.

MATHEMATICS AND MORALS.*

By C. H. Gore, M.A.

I fear you may think us rather unpractical people in Hull, for while my colleague, Mr. Forder, will presently speak to you on the subject of infinity, my own topic has not a little of the unreal or imaginary involved in it—but at least, if I become imaginary and Mr. Forder tends to infinity, we shall not, like 'I' and 'J,' be both at infinity and unreal at once. My own thesis is not a new one, for it is at least as old as Plato. It is a claim for greater attention to the more beautiful aspects of mathematics. What one feels is that most people give to Mathematics and Science very definite and, as I think, very limited spheres of influence in the educational domain. In their view it is the duty of school science work to beget accuracy of observation, skilful manipulation and the power to make general deductions from observed facts—it is an essentially materialistic and utilitarian view. As a matter of fact, Natural Science deals with the profoundest mysteries. So, with them, Mathematics is a science cold and austere—ruthlessly precise and devoid of imagination. And yet, as a matter of fact, it deals with truth and beauty, its forms are those of rhythm and symmetry, and in it imagination has no limits. Now I fear very much that we who teach these subjects are apt to accept without demur these low valuations of mathematics and science as means of general education. Yet in Chemistry, Physics, Geology and Biology there is ample scope for the training or development of this sense of wonder and of reverence. Less than half a century ago we believed that civilisation was to be saved by teaching our boys practical science, which could be applied usefully and which would make for commercial prosperity. The products of this system knew everything—or thought they did. This world, they said, was nicely filled up with atoms and molecules, and what more could you possibly want? The study of the humanities to these people was a waste of time—poetry, art, and literature of little account. They had 'done' science and were horribly cock-sure about it all. And yet what a difference when we consider men who have carried their science far. Our great scientists are of all men the most modest and humble—their knowledge is to them only the fringe of a far greater unknown. Space, Time, and Life are their trinity of mysteries: they are moved by an irresistible and insatiable curiosity about the hidden truths of Nature, and they have learnt that Truth can only be found by truthfulness and knowledge by humility. And just in the same way a boy's curiosity about the world in which he lives is insatiable: he wants to know. Hence possibly the Heuristic Method! But does even that method satisfy the boy's longings? I think most certainly not. He wants to know about wonderful things like motor engines, wireless, high explosives, dynamos, and we set him to work to measure the area of a triangle or the volume of a cylinder. His ardour is damped. This sort of thing has to be done no doubt, and manipulative skill is an important part of science training, but I could wish there were more lessons given in which the marvels and wonders of nature were the subject. We cannot look at the stars or think of their distances or their ages without awe and reverence. A single drop of water is a miracle. But we do not speak of these things. Can we be surprised, then, that one of the chief characteristics of young people to-day is their lack of reverence? The particular kind of morality which science could cultivate is just that sense of reverence. When we approach the study of Mathematics we meet with two obvious views or aspects of the subject, neither of which seems to me to be quite satisfying. First, Mathematics as a deductive logical process of the

*A paper read at a meeting of the Yorkshire Branch at Hymers College, Hull, June 17, 1922.

pure reason—unmoral and soulless—secondly, Mathematics as a sort of accountant to the firm of Physics & Co., whose duty it is to provide formulas and do integrations which may smooth the path of the scientific explorer. This, of course, is the more modern utilitarian view of mathematical training: under it our geometry text-books have passed from logic to mensuration, while our trigonometries are concerned chiefly with apparatus and utensils. Now I think there is another point of view which we are in grave danger of neglecting. Mathematics is surely the study of the symbolic representation of order and rhythm. In no other study can we so simply and so definitely exhibit the beauty of order or the real meaning of rhythm. No subject, unless it be music, can give us the feeling that we are dealing with eternal verities as can Mathematics. Absolute truth and honesty are necessary in its processes—the less it is mixed up with mundane considerations the more perfect and beautiful are its results. An error spoils the picture or may even wreck the whole of the work. In Mathematics to sin is unpardonable, or, at least, pardon can only be achieved by renewed labour and toil. Every step of the work must be true, cannot be scamped. We may guess and make our rough checks or approximations, but all the time we have only the one goal in view. The discovery of the perfect solution gives a feeling of satisfaction akin to holing out your drive in one at golf, or hooking a forty-pound salmon, or discovering a perfect melody. There is a laughter of sheer joy which comes in mathematical class-rooms (perhaps rarely), but which I believe is only to be heard there—the pure laughter of the gods when some new law has been discovered. It is not laughter at the comic, but laughter at the beautiful. And so it is that the mathematician often comes to be a simple-minded person whose pleasures are not derived from human eccentricities, romances, or tragedies, but rather from abstract beauty and perfect rhythm. In his soul there is a deep sense of divine order and a profound belief in universal law. Fortunately, perhaps, we are not all such perfectly pure mathematicians as to reach this level, but at least we have something of this in our constitution. We may quote Plato extensively to bear out this view. In the *Republic* (iii. 401) he says, "Musical training is a more potent instrument than any other, because rhythm and harmony find this way into the secret places of the soul, on which they mightily fasten, imparting grace and making the soul graceful of him who is rightly educated or ungraceful of him who is ill-educated: and because he who has received this true education of the inner being, will most shrewdly perceive omissions or faults in art and nature." The true appreciation of virtue, the critical power to discriminate between the good and the bad, between the beautiful and the unbeautiful—the moral sense, in fact—is developed subconsciously by the study and practice of things rhythmical and symmetrical. Thus the expression of the sine of an angle in terms of its circular measure $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \ldots$ is rhythmical and beautiful, and makes for good morals. The formula, $t = 2\pi\sqrt{l/g}$ is materialistic and has nothing to do with morality! It therefore comes about that we may, by stressing the things beautiful in Mathematics, help to ennoble the moral sense, and it also comes about that by dealing chiefly with the practical side of Mathematics we shall lessen this good influence. I am quite sure that this is not a fantastic view. There are many excellent persons who think you can teach morality by maxims or by Bible texts. The moral sense is a far more delicate instrument than that—it is to a great extent subconscious. We have an instinctive consciousness of the standard of right. We have another set of standards based on social conventions, and another on pragmatical necessities. We are beginning to learn much about these subconscious regions of the mind and soul, and we have passed, I hope, from the limited view that the subconscious is merely a region of repressed sex and other primary complexes. There is room also for the good. Plato insists again and again that

Music and Mathematics (pure Mathematics) can by their rhythm and harmony fill the soul with beauty, and make it a fount and source of pure and well-balanced thought and ideals. My contention is that the study of Pure Mathematics develops subconsciously a sense of order, law, and rhythm which may easily so foster the critical instinctive sense that right thinking and right judgment—morality in fact—become a natural growth of the mind. Good Music has this result most certainly. But then Music and Mathematics have very much in common: both have rhythm and periodicity, both have contour and form. A boy of thirteen, who was just beginning graphs in Algebra and is also a very keen musician, brought to me the other day a movement of Beethoven turned into a group of graphs. It was his own notion: he had grasped the connection between music contour and graphs without a hint from anyone! He has since done a fugue of Bach's which proved to be a much more suitable subject for the experiment. Some of you may have seen a book by Mrs. Arthur Somervell called *A Rhythmic Approach to Mathematics*, in which children are taught to get curve envelopes by drawing many successive positions of straight lines obeying some law, *e.g.* the falling ladder, the dog running after his master across a field (the curve of pursuit). Also, I think these are examples of the Tractrix, and so on. Children literally 'go mad' about these things: they make up designs from them, and love to colour the panels between successive lines (some of these designs were exhibited). Occasionally interesting discoveries are made: thus a boy of thirteen discovered that if a chord of a circle moves so that one end travels round the circle twice as fast as the other end, the envelope of this chord is the cardioid curve. The discovery, which was new to me, produced quite an epidemic of cardioids in the Third Forms. I do not suppose this sort of thing is, for your practical mathematician, of great service, but it is extraordinarily stimulating in a general way, and in particular increases, I think, the subconscious appreciation of the fact that law and order are essentials of the beautiful.

Thus we may consider any geometrical Theorem in three distinct ways: (1) as an experimental piece of work to be tested by actual drawing, (2) as a logical process of pure reason, or (3) as a result of real intrinsic beauty. I think this last point of view should never be ignored: it comes equally well from either the first or the second way of considering the proposition. It produces a feeling that the result is complete and perfect and satisfying. In elementary Algebra the graph is a fertile instrument of appeal to this sense of the fit and suitable. The simpler the function the simpler the graph. A reasonable looking equation does not produce a bumpy looking graph. All sorts of fascinating things may be tried. A class having done a few examples in Arithmetic or Algebra may be made to draw the frequency curve of their marks—the class soon appreciates that the hump on the curve should not be too far to the left. I have been asked by a class to allow them to do more sums in order that they might improve the shape of their frequency graph! Truly a notable example of the moral influence of Mathematics!

Again, the idea of growth as illustrated by an orderly curve and not by a feverish temperature chart or a statistical graph of, say, imports or exports, conveys the feeling that the real good is reached slowly and by imperceptible steps. Things beautiful and moral grow in our minds by orderly curves of progress just as the Kingdom of Heaven itself is a mysterious growth in a man's soul. If $\frac{dy}{dx}$ is positive, we have the Kingdom of Heaven; if $\frac{dy}{dx}$ is zero, then there is no growth and no kingdom, and if it is negative, I fear we can only think of another sort of kingdom better not named.

The ideal can only be reached asymptotically, and yet it is worth while trying for it all the time. Morality is just the travelling along a curve which stretches out to touch the asymptotic ideal at infinity. And here we can imagine

many curves, some closer, some further away than our own, all tending to the same ultimate limit. Thus, instead of regarding morality as a code, we come to feel it as a path of progress to the ideal. We can measure both the distance covered and the ever diminishing separation from the ultimate perfection. Mathematics and History have this in common that both treat of the development of certain variables in time. The great curves in History can be traced over wide periods, and show ultimate law and progress, but in detail they are blurred or distorted by the obstinacies or stupidities of men, just as the statistical or observational graphs of the man of science are distorted by inaccuracy of measurement. Now our mathematical curves are pure and undefiled. They satisfy the highest canons of taste, the progress law of morality, and the instinctive desire of the mind for beauty. They are divine.

Two suggestions remain: First of all we must not perpetually call attention to the symmetrical or the beautiful in Mathematics: it must be allowed to soak in: it will so bear its best fruit. There will be lessons in which it appears not at all, but at least it can often appear, and some lessons may be devoted almost entirely to the rhythmic order and symmetry of mathematical ideas. Secondly, it can be introduced at the most elementary stages, as I shall presently show, and in this respect it seems to me that Mathematics differs entirely from language work, where the elementary stages are practically devoid of anything suggestive of moral beauty—I am not, of course, speaking of English, but of foreign tongues when the structure of the language and the vocabulary have to be learnt. If a boy learns Latin from twelve to sixteen, I doubt if he gains any great reward in such things as taste and feeling for the fit and beautiful—he has had a training in logical reasoning and in accuracy, but he must continue his work beyond sixteen if he is to get the real advantages which come from the study of the Classics. But you can use the multiplication table with children of eight or nine or less to introduce the first glimmer of the idea of law and periodicity: thus 7 times with its digits added gives

	7	14	21	28	35	42	49	56	63	70
	7	5	3	1	8	6	4	2	9	7
4 times,	4	8	12	16	20	24	28	32	36	40
	4	8	3	7	2	6	1	5	9	4

Again, congruences delight quite small children. Take the 5 family, $5n$ is the father, $5n+1$ the eldest son, $5n+2$ the second son, $5n-1$ the eldest daughter, $5n-2$ the younger, and so we get a series of family groups living next door to one another:

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22

The square numbers are always fathers or elder children. Is 327 a son or a daughter? Why can it not be a square?

But you will think of many other things which are suggestive of symmetry and beauty in dealing with numbers.

In conclusion, my contention is merely that any subject of study which is rhythmic, orderly, and symmetrical, produces in the mind a love and desire for all things beautiful, and provides the subconscious regions of the soul with running springs of living harmony which may well act as solvents to the dreadful complexes with which the psychologists terrify and threaten us. My plea is that we should not neglect these things in Mathematics—I ask you not only to make it the training ground for accuracy in calculation and measurement, and for strictness and honesty in logical processes, but also to allow it to inspire unobtrusively and unconsciously that sense of order and rhythm and beauty which is the foundation of all progressive morality.

C. H. GORE.

APPROXIMATE INTEGRATION.*

By Arnold Buxton, M.A.

I. The process known as the differentiation of functions in the Calculus does not present the insurmountable difficulties one experiences in attacking the converse problem associated with the Integral Calculus. As is well known, there are many functions which we cannot integrate, and as the process of integration is really a summation, this means there are many areas which cannot be evaluated by exact methods. There are many cases in various branches of science in which it is required to determine the area of a closed curve obtained by plotting a series of observations. The engineer, for example, has to determine the areas of indicator and similar diagrams, the naval architect has to calculate volumes, and centres of gravity or buoyancy by approximate methods. The mathematician may be called upon to evaluate definite integrals which cannot be integrated directly, such as the elliptic integrals. The problem reduces simply to the consideration of the determination of the area between the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_0+nh$, and the axis of x; and the first part of the paper will be devoted to the methods of obtaining empirical rules for approximation to the evaluation of such an area when $f(x)$ cannot be directly integrated. In the second part, the mechanical methods of integration will be considered, including demonstrations with the different types of instruments illustrating these methods.

II. *To obtain simple empirical rules for approximate integration.*

(*a*) In Fig. 1, if the curve PQ is fairly flat, an easy approximation is obtained by putting $y=A$, where A is a constant.

Then $$\text{Area } PQQ_0P_0 = \int_{x_0}^{x_0+nh} A\,dx = Anh = y_0 \,.\, P_0Q_0.$$

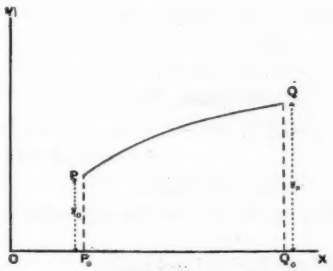

Fig. 1.

If the curve is not flat, the large error introduced can be lessened by taking a large number of ordinates between $x=x_0$ and $x=x_0+nh$ and applying the method to each rectangle, so that the expression for the area now becomes

$$\int_{x_0}^{x_0+nh} f(x)dx = y_0h + y_1h + y_2h + \ldots + y_{n-1}h,$$

* A paper read to the London Mathematical Association, Nov. 16, 1921.

where the length P_0Q_0 is divided into n equal parts and $y_0, y_1, y_2, \ldots y_n$ are the corresponding ordinates, or

$$\int_{x_0}^{x_0+nh} f(x)dx = y_1h + y_2h + \ldots + y_nh,$$

starting at the second ordinate first and finishing at y_n.

The two approximations so obtained are known as the Rectangular Rules, and are used in the so-called Harrison's method in Harmonic Analysis.

(*b*) Having obtained empirical rules by considering y = a constant, A, the next method is clearly to obtain rules for $y = A + Bx$.

Then
$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+nh} (A+Bx)dx = \left[Ax + \frac{Bx^2}{2} \right]_{x_0}^{x_0+nh}$$
$$= nh[A + B(x_0 + x_0 + nh)/2]$$
$$= nh[A/2 + Bx_0/2 + A/2 + B(x_0 + nh)/2]$$
$$= nh[y_0 + y_n].$$

By dividing the length P_0Q_0 into n equal parts and applying this approximation to each trapezium so formed, the formula becomes

$$\text{Area} = \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})].$$

This rule is commonly called the Trapezoidal Rule, and is clearly the arithmetic mean of the two Rectangular Rules.

(*c*) Having dealt with $y = A$, $y = A + Bx$ as possible forms of the curve, the next method is to try $y = A + Bx + Cx^2$. Such a curve (parabola) is given when three ordinates are given. Hence, taking the three equidistant ordinates $x = x_0 - h$, $x = x_0$, $x = x_0 + h$, it follows that

$$\int_{x_0-h}^{x_0+h} f(x)dx = \left[Ax + Bx^2/2 + Cx^3/3 \right]_{x_0-h}^{x_0+h}$$
$$= A \,.\, 2h + 2Bx_0h + C(6x_0^2h + 2h^3)/3$$
$$= 2h[6A + 6Bx_0 + 6Cx_0^2 + 2Ch^2]/6$$
$$= h[A + B(x_0 - h) + C(x_0 - h)^2 + 4\{A + Bx_0 + Cx_0^2\} + A + B(x_0 + h) + C(x_0 + h)^2]/3$$
$$= h[y_0 + 4y_1 + y_2]/3.$$

By repeated application of this rule, the expression for the area becomes equal to

$$h[y_0 + 4y_1 + y_2 + y_3 + 4y_3 + y_4 + \ldots + y_n]/3$$
$$= h[y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]/3,$$

where n is even.

This is of course the well-known Simpson's Rule.

By taking a polynomial of the third degree, $y = A + Bx + Cx^2 + Dx^3$ and four equidistant ordinates, we can similarly evolve another approximate formula, the accuracy of which should be correspondingly increased by repeated applications. The integration of the polynomial of the 4th and higher degrees is easy enough, but the subsequent rearrangement to fit in the ordinates becomes very involved and cumbersome. Hence it is necessary to consider a more general method of obtaining the empirical rules.

III. *To obtain a general formula to include all empirical rules with equidistant ordinates.*

It has appeared so far that the main feature of these empirical formulae is to assume the area as equal to $B_0y_0+B_1y_1+B_2y_2+\ldots+B_ny_n$, where $B_0, B_1, B_2, \ldots B_n$ are constant multiples of the interval h. In the case of Simpson's Rule, for example, where $n=2$, $B_0=h/3$, $B_1=4h/3$, $B_2=h/3$. Bearing this form in mind, we can now proceed to the general formula. Let

$$F(x)=(x-\alpha_1)(x-\alpha_1-h)(x-\alpha_1-2h)\ldots(x-\alpha_1-nh)$$

define the position of n ordinates equally spaced from $x=\alpha_1$; then $f(x)$, the approximation curve, through these ordinates, will be of the nth degree. Hence, by a theorem in Partial Fractions,

$$\frac{f(x)}{F(x)}=\sum_{r=0}^{r=n} f(\alpha_1+rh)/F'(\alpha_1+rh)\times 1/(x-\alpha_1-rh),$$

i.e.
$$f(x)=\sum_{r=0}^{r=n} f(\alpha_1+rh)/F'(\alpha_1+rh)\{F(x)\}/(x-\alpha_1-rh),$$

and
$$\int f(x)\,dx=\sum_{r=0}^{r=n} B_r f(\alpha_1+rh), \text{ where } B_r=\int F(x)\,dx/F'(\alpha_1+rh)(x-\alpha_1-rh),$$

$$=\sum_{r=0}^{r=n} B_r y_r,$$

i.e.
$$\int f(x)\,dx=B_0y_0+B_1y_1+B_2y_2+\ldots+B_ry_r+\ldots+B_ny_n.$$

Thus we have an expression for the area in the usual form, and it remains to evaluate the coefficients $B_0, B_1, \ldots B_n$. It is worth while noticing that these coefficients denote the weight that should be given to the successive ordinates. Taking the evaluation for $n=2$,

$$B_2=\int_{(\alpha_1)}^{\alpha_1+2h}(x-\alpha_1)(x-\alpha_1-h)\,dx/2h\,.\,h$$

$$=\left[\frac{x^3}{3}-\alpha_1x^2-\frac{hx^2}{2}+\alpha_1{}^2x+\alpha_1hx\right]_{\alpha_1}\Big/2h^2$$

$$=[3\alpha_1{}^2+4h^2+6h\alpha_1-6\alpha_1{}^2-6\alpha_1h-3h\alpha_1-3h^2+3\alpha_1{}^2+3\alpha h_1]\div 3h$$

$$=h/3.$$

$$B_1=\int_{(\alpha_1)}^{\alpha_1+2h}(x-\alpha_1)(x-\alpha_1-2h)\,dx/(h)(-h)$$

$$=4h/3.$$

Similarly

$$B_0=h/3.$$

Also by putting $n=3$, and evaluating B_3, B_2, B_1, B_0, we obtain Simpson's Three-eighths Rule.

Moreover, $n=1$ gives the Trapezoidal Rule coefficients;

$n=4$ gives $2h(7y_0+32y_1+12y_2+32y_3+7y_4)/45$,

a rule given in Dr. Sheppard's article on "Mensuration" in the *Encyclopaedia Britannica* and used extensively in the writer's paper on "Star-Discs" in the *Monthly Notices, R.A.S.*, June, 1921. $n=6$ gives Weddle's Rule,* and more

* This statement is not strictly true, as one of the coefficients has been slightly altered round off the numbers.

can be manufactured by putting $n=5, 7, 8, 9$, etc. This general theorem can also be obtained by the integration of Newton's interpolation formula

$$y_m=(1+\Delta)^m y_0=y_0+m\Delta y_0+{}^mC_2\Delta^2 y_0+\ldots+{}^mC_n\Delta^n y_0,$$

where the nth differences are constant. In other words, the function y is assumed to be a polynomial of the nth degree, so that the weighting factors of the ordinates B_0, B_1, B_2, ... B_n must have the same values as before.

IV. *To obtain a general theorem when the ordinates are chosen.*

Hitherto we have considered only equidistant ordinates, and the question naturally arises as to whether we can choose our ordinates in order to obtain closer approximations. In this connection, it is important to note that Gauss has a theorem that, by properly choosing n ordinates, the same accuracy can be obtained as with $2n-1$ equally spaced ordinates. As before, let

$$F(x)=(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$$

define the positions of the n chosen ordinates $x=a_1, a_2, \ldots a_n$; then the approximation curve $f(x)$ through the extremities of these n ordinates is of degree $n-1$. Now let the curve $\phi(x)$ pass through the extremities of $(2n-1)$ equally spaced ordinates defined by $x=\alpha_1, \alpha_1+h, \alpha_1+2h, \ldots \alpha_1+\overline{2n-1}h$; then $\phi(x)$ will be of the $(2n-1)$th degree, and we are at liberty to arrange it to pass through the extremities of the n chosen ordinates. Then we can write

$$\phi(x)-f(x)=M\times F(x),$$

since both sides vanish for values of x, a_1, a_2, ... a_n, and M will therefore be a function of x of the $(n-1)$th degree. Integrating the above expression between the limits q $(=\alpha_1)$ and p $(=\alpha_1+\overline{2n-1}h)$, we have

$$\int_q^p \phi(x)dx-\int_q^p f(x)dx=\int_q^p M\times F(x)dx.$$

Integrating the right-hand side (R.H.S.) by parts,

$$\text{R.H.S.}=\Big[F_1(x)\,.\,M-F_2(x)M'+F_3(x)M''-\ldots+(-1)^{n-1}F_n(x)M^{n-1}\Big]_q^p,$$

$F_r(x)$ denoting the rth integral of $F(x)$

and M^{r-1} ,, ,, $\overline{r-1}$th differential of M.

Since M is of the $\overline{n-1}$th degree, then clearly M^{n-1} is a constant. Hence we are in a position now to investigate the form of $F(x)$, and thereby determine the positions of the chosen ordinates, for we have only to make the R.H.S. equal to zero or to choose $F(x)$ so that $F_1(x)$, $F_2(x)$, ... $F_n(x)$ vanish at the limits. Then we shall have

$$\int_q^p \phi(x)dx=\int_q^p f(x)dx.$$

Further, having determined a_1, a_2, a_3, ... a_n we shall then be able to write the expression for the area as in III., $B_1f(a_1)+B_2f(a_2)+\ldots+B_nf(a_n)$, where B_1, B_2, ... B_n are the new coefficients to be determined.

The condition that $F_1(x)$, $F_2(x)$, $F_3(x)$, ... $F_n(x)$ should vanish at the limits is clearly fulfilled when

$$F(x)=\frac{d^n}{dx^n}\{(x-p)^n(x-q)^n\}.$$

Now a_1, a_2, a_3, ... a_n are the n roots of $F(x)=0$, and are therefore the roots of

$$\frac{d^n}{dx^n}\{(x-p)^n(x-q)^n\}=0.$$

By the simple substitution $x=\frac{1}{2}(p+q)+\frac{1}{2}(p-q)y$, this reduces to

$$\frac{d^n}{dy^n}(y^2-1)^n=0 \quad \text{or} \quad P_n(y)=0,$$

where $P_n(y)$ is the Legendre Function of the nth order, and whose roots are tabulated for numerical values of n in either Jahnke and Emde (*Funktionentafeln mit Formeln und Kurven*), or in Dale's *Mathematical Tables*.

Now let the curve $f(x)$ become by the above substitution $\psi(y)$; then, since the limits of y are -1 and 1, and by the Theory of Partial Fractions as in III.,

$$\int_{-1}^{1}\psi(y)dy=\sum_{r=1}^{r=n}\psi(a_r)/P_n'(a_r)\int_{-1}^{1}\{P_n(y)/(y-a_r)\}dy$$
$$=\Sigma B_r\psi(a_r)=B_1y_1+B_2y_2+B_3y_3+\ldots+B_ny_n,$$

where $y_r=\psi(a_r)$ and $B_r=\int_{-1}^{1}[P_n(y)/P_n'(a_r)(y-a_r)]dy$.

If $n=3$, for example, we have to evaluate B_3, B_2, B_1. Now $P_3(y)=0$ has the three roots, 0, $\pm\sqrt{\frac{3}{5}}$ (α_1, α_2, α_3), so that

$$B_3=\int_{-1}^{1}\{(y+\alpha_2)y/2\alpha_2^2\}dy=5/9,$$

$$B_2=\int_{-1}^{1}[(y+\alpha_3)y/2\alpha_2^2]dy=5/9$$

and

$$B_1=\int_{-1}^{1}[y+\alpha_2)(y+\alpha_3)/(-\alpha_1^2)]dy=8/9.$$

Hence

$$\int_q^p f(x)dx=\tfrac{1}{2}(p-q)\int_{-1}^{1}\psi(y)dy=\tfrac{1}{2}(p-q)[(5/9)\psi(\alpha_2)+(5/9)\psi(\alpha_3)+(8/9)\psi(\alpha_1)].$$

Now $\psi(\alpha_2)=f(p+q)+\overline{(p-q}\,\alpha_2)/2$, for instance,

Therefore

$$\int_q^p f(x)dx=(p-q)[(5/9)f(c_1)+(8/9)f(c_0)+(5/9)f(c_1')]/2,$$

where

$$c_0=(p+q)/2,$$
$$c_1=(p+q)/2+(p-q)\alpha_2/2=(p+q)/2+(p-q)\sqrt{3/5}/2,$$
$$c_1'=\tfrac{1}{2}(p+q)+\tfrac{1}{2}(p-q)\alpha_3=\tfrac{1}{2}(p+q)-(p-q)\sqrt{3/5}/2.$$

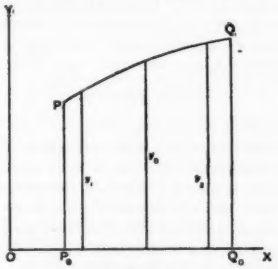

FIG. 2.

If the curve be given as in Fig. 2, for example, we have to measure

(1) the ordinate midway between PP_0 and QQ_0 (y_0, say);

(2) the ordinates at a distance $\cdot 3873 P_0Q_0$ on either side of y_0 (y_1 and y_2, say).

Then $$\text{Area} = (P_0Q_0/2)[5(y_1+y_2)+8y_0]/9.$$

When $n=4$ or 5 the B-coefficients can be evaluated in a similar manner, remembering that in the latter case, for example, $P_5(y)=0$ gives

$$y(y-\alpha_1)(y+\alpha_1)(y-\alpha_2)(y+\alpha_2)=0,$$

where $\alpha_1=\cdot 5382$, $\alpha_2=\cdot 9061$, so that the corresponding formula becomes

$$\int_q^p f(x)dx=(p-q)[\cdot 2841 f(c_0)+\cdot 1184\{f(c_1)+f(c_1')\}+\cdot 2393\{f(c_2)+f(c_2')\}],$$

where $$c_0=(p+q)/2,$$
$$c_1=(p+q)/2+(p-q)\alpha_1/2, \qquad c_2=(p+q)/2+(p-q)\alpha_2/2,$$
$$c_1'=(p+q)/2-\tfrac{1}{2}(p-q)\alpha_1/2, \quad c_2'=(p+q)/2-(p-q)\alpha_2/2,$$

or graphically,

$$\text{Area}=P_0Q_0[\cdot 2841 y_0+\cdot 1184(y_1+y_2)+\cdot 2393(y_3+y_4)],$$

where y_0 is the ordinate midway between PP_0 and QQ_0,
and y_1 and y_2 are the ordinates distant from P_0Q_0 on either side of it,
and y_3 and y_4 " " " " " " " "

Other rules can be obtained simply by varying n. Those for $n=4$ and $n=7$, together with the analysis for the values of the coefficients $B_1, B_2, \ldots B_n$ in the general case, are to be found in *British Association Report*, 1919, "On Gauss's Theorem for Quadrature and the Approximate Evaluation of Definite Integrals with Finite Limits," by Professor A. R. Forsyth, F.R.S.

The above rules were applied to $\int_1^7 dx/x$ or $\log_e 7$ with the following results:

		Error.
Trapezoidal Rule applied six times - -	2·01243	·07552
Simpson's " " three " - -	1·95873	·01282
Three-Eighths " " twice - -	1·96607	·02016
Weddle's Rule - - - - - -	1·95286	·00695
Gauss' Three Ordinate Rule - - -	1·9323	−·0136
" Five " " - - -	1·95048	·0045
Correct value - - - - - -	1·94591	

Other results dealing particularly with Prof. Forsyth's rules from the Gaussian theorem are given in the *B.A. Report* already referred to. The seven-chosen-ordinate is equivalent to the rule obtained by considering 13 equidistant ordinates, and so gives as high an accuracy as can be expected. Just lately I have compared the Gaussian three-ordinate-rule against the 4 equidistant-ordinate-rule on $J_0(x)$ for values of x from 0° to 150° for every 10° and working to the three decimal places. Comparing the two sets of results with the tabulated values of $J_0(x)$, there was very little difference in the results.

(*To be continued.*)

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142. . . . . Those that know the mathematic lines
Where Nature all the wit of man confines,
And when it keeps within its bounds, and where
It acts beyond the limits of its sphere
Enjoy an absoluter free command . . . .
Will undertake the universe to fathom,
From infinite down to a single atom,
Without a geometric instrument. . . .

" *Fragments on the abuse of Human Learning*,"—
Butler's *Works*, ii. p. 229 (Aldine edition).

SOME INCIDENTAL WRITINGS BY DE MORGAN.

(*Continued from* p. 149, vol. x.)

XX.

I. ii. 199. **Logic.**—What is the earliest printed book on logic? meaning the first which gives the common theory of the syllogism. Does it contain the celebrated words *Barbara, Celarent,* &c. ? The difficulty will probably arise from this, that each book has some *undated* editions which are probably earlier than the dated ones. Of books with dates there is the exposition of Petrus Hispanus by Joh. Versor, in 1473, and the *Summulae* of Paulus Venetus, in 1474; the first I find in Hain (who had not seen it), the second I have seen. Can any one of your readers go further back? M.

[Ludwig Hain (*Repertorium Bibliographicum*), 1826-1838, Stuttgart, indices by Conrad Berger, 1891, 1908. The *N.E.D.* gives 1589 (*Marprel. Epit.* E iiij b) as the earliest date for *Barbara* in English. *Barbara* (Lat.) = "barbarous things." The word is a mnemonic (the three a's in Barbara), for "A" indicates a universal affirmation.

The first text-book on logic written in English was Thomas Wilson's *The Rule of Reason*, 1551, 1552, 1554. Wilson was an Etonian and Master of St. Katherine's Hospital by the Tower. The 3rd edition gives the only evidence of the authorship of Roister Doister, for the title page of the only known copy of the "interlude made by Nicholas Udal" is missing, and Wilson quotes from this ancient play a letter in which is an "ambiguitie" (*Encycl. Brit.* "Udal," xxvii. 555).

The *Dialectique* of Ramus, 1555, is the first instance of a French departure from the old scholastic tradition of composing all philosophical works in Latin (*loc. cit. French Literature* xi. 126; *Ramus*, xxii. 881).]

XXI.

I. ii. 263. **"Crede quod habes," etc.**—The celebrated answer to a Protestant about the real presence, by the borrower of his horse, is supposed to be made since the Reformation, by whom I forget:

"Quod nuper dixisti
De corpore Christi
Crede quod edis et edis;
Si tibi rescribo
De tuo palfrido
Crede quod habes et habes."

But in Wright and Halliwell's *Reliquiae Antiquae*, p. 287, from a manuscript of the time of Henry VII., is given:

"Tu dixisti de corpore Christi, crede et habes
De palefrido sic tibi scribo, credo et habes."

M.

XXII.

I. ii. 266. **Bernardus Patricius.**—Some writers mention *Bernardus* Patricius as a follower of Copernicus, about the time of Galileo. Who was he? M.

[Was this Francesco Patrizzi (Patrizio), Franciscus Patritius (1529 (? 1530)-1597), who was a Copernican? If so *v. Encyc. Brit.* "Patrizzi," xx. p. 934, and *Bayle*, "Patrice," pp. 749-750, note (B). Edition 1697, Rotterdam.]

XXIII.

I. ii. 296. **Engelman's Bibliotheca Scriptorum Classicorum.**—A little while ago, I ordered Engelmann's *Bibliotheca Scriptorum Classicorum*, purporting to contain all such works published from 1700 to 1846. It was furnished to my bookseller by a foreign bookseller *in London*, with an English title, having *his own* name on it as publisher, and an invitation to purchase the books described in it *from him*. As the paper and type were German, I objected; and received in consequence a new English title, with the same name upon it, and a *shorter* invitation to purchase from him. I was captious enough to object even to this; and I then received a Leipzig title in German. But there still remains a difficulty: for this German title has also the name of a *Parisian* bookseller upon it, *à la maison duquel on peut s'adresser*, &c. Now, as Engelman is a bookseller, and would probably not object to an order out of his own catalogue, of which he is both author and publisher, the preceding circumstances naturally raise the following Queries: 1. What is the real title page of Engelman's *Catalogue*? 2. Is the Parisian house accredited by Engelman; or has the former served the latter as the London house has served both? 3. Is it not desirable that literary men should set their faces very decidedly against all and every the slightest alteration in the genuine description of a book? 4. Would it not be desirable that every such alteration should forthwith be communicated to your paper?

The English title-page omits the important fact, that the *Catalogue* begins at 1700, and describes it as containing *all* editions, &c., up to 1846. A. De Morgan.

The next week the following communication was forthcoming from the champion of the booksellers:

(P. 312.)

The shortest reply to Mr. De Morgan's complaint against a foreign bookseller would be, that *Engelmann himself* printed for any of the purchasers of a large number of his Catalogues the titles to which Mr. De Morgan objects so much.

Will you allow me to add one or two remarks occasioned by Mr. De Morgan's strictures?

1. Engelmann is not, strictly speaking, a bookseller, and his catalogues are not booksellers' catalogues in the sense in which that term is generally received here. He is a publisher and compiler (and an admirable one) of general classified catalogues for the use of the trade and of students, without any reference to his stock, or, in many instances, to the possibility of easily acquiring copies of the books enumerated: and although he *might* execute an order from his catalogues, getting orders is *not* the end for which *he* publishes them.

2. Some foreign houses in London, as well as in other countries, bought a large number of his Catalogues, not as a *book* but as a *catalogue*, to be supplied to their customers at the bare cost, or, where it appears advisable, to be delivered gratis to purchasers of a certain amount.

3. It appears to me pardonable, if, under these circumstances, a notice is inserted on the title, that orders may be directed to the house which has purchased a number, and supplies them without any immediate profit; and I may add that I do not believe any of the houses concerned would object to a notice being taken of such a proceeding in your paper.

4. The error in omitting the words "from 1700" on the title page, is one to which Mr. De Morgan's notice first directed my attention, classics printed before that date not being commonly in demand among foreign booksellers.

5. The practice of compiling catalogues for general use, with the names of the purchasers of any number of copies of the catalogue inserted on the title or wrapper, is very common in Germany.

Hinrichs of Leipsic issues :

1. A Six-monthly Alphabetical Catalogue, with a systematic index ;
2. A Quarterly Catalogue, systematically arranged, with an alphabetical index.

Vandenhoeck of Gottingen issues *half-yearly* :

1. A Bibliotheca Medico-Chirurgica et Pharmaceuto-Chemica ;
2. A Bibliotheca Theologica, for Protestant theology ;
3. A Bibliotheca Classica et Philologica ;
4. A Bibliotheca Juridica ;

and Engelmann, from time to time, numerous general catalogues ; all of which are not only supplied to London houses, with English titles, but may be had all over Germany, with the firms of different booksellers inserted as publishers of the catalogue.

Will you make use of the above in any way in which you may think it of advantage to your readers ? ANOTHER FOREIGN BOOKSELLER.

This drew the following reply (p. 328) : The sort of defence, explanation, or whatever it may be called, founded upon usage, and offered by ANOTHER FOREIGN BOOKSELLER, is precisely what I wanted to get out, if it existed, as I suspected it did.

If your correspondent be accurate as to Engelmann,* it appears that no wrong is done to *him* ; it is only the public which is mystified by a variety of title-pages, all but one containing a suppression of the truth, and the one of which I speak containing more.

I now ask you to put in parallel columns extracts from the title given by Engelmann with the substitutes given in that which I received.

"Schriftsteller-welche vom Jahre 1700 bis zu Ende des Jahres 1846 besonders in Deutschland gedruckt worden sind."	Classics . . . that have appeared in Germany and the adjacent countries up to the end of 1846.

I do not think it fair towards Mr. Engelmann, whose own title is so true and so precise, to take it for certain, on anonymous authority, that he sanctioned the above paraphrase. According to the German, the catalogue contains works from 1700 to 1846, published *especially* in Germany ; meaning, as is the fact, that there are some in it published elsewhere. According to the English, all classics printed in Germany, and all the adjacent countries, in all times, are to be found in the catalogue. I pass over the implied compliment to this country, namely, that while a true description is required in Germany, a puff both in time and space is wanted for England. I dwell on the injurious effect of such alterations to literature, and on the trouble they give to those who wish to be accurate. It is a system I attack, and not individuals. There is no occasion to say much, for publicity alone will do what is wanted, especially when given in a journal which falls under the eyes of those engaged in research. I hope those of your contributors who think as I do, will furnish you from time to time with exposures ; if, as a point of form, a Query be requisite, they can always end with, Is this right ? A. DE MORGAN.

The next stage is an affirmation by Herr Engelmann (p. 397) :

I hereby attest that the English titles to my *Bibliotheca Scriptorum Classicorum* were *not* printed without my knowledge or wish, but *by myself*, for my customers in England. W. ENGELMANN.

Leipzig, Oct. 25, 1850.

I also enclose the original, for the benefit of Mr. de Morgan, if he is not satisfied. ANOTHER FOREIGN BOOKSELLER.

[It will be seen that these "replies" ignore completely the definite charge brought by Mr. De Morgan, so that he had no particular reason to be satisfied by this enclosure.]

* Here De Morgan adopts the name "Engelmann."

XXIV.

I. ii. 317. **Aerostation.**—The article BALLOON, in the *Penny Cyclopaedia*, would give C.B.M. a good many references. The early works there mentioned are those of Faujas de St. Fond, Bourgeois, and Cavallo; to which I add the following: Thomas Baldwin, *Airopaidia, containing the Narrative of a Balloon Excursion from Chester, Sept.* 8, 1785, Chester, 1786, 8vo (pp. 360).

Vincent Lunardi published the account of his voyage (the first made in England) in a series of letters to a friend. The title is torn out in my copy. The first page begins, "An Account of the First Aerial Voyage in England. Letter I. London, July 15, 1784." (8vo, pp. 66 + ii., with a plate.) It ends with a poetical epistle to Lunardi by "a gentleman well known in the literary world" (query, the same who is cited thus in our day?) from which the following extracts are taken as a specimen of the original balloon jokes:

"The multitude scarcely believed that a man,
With his senses about him could form such a plan,
And thought that as Bedlam was so very nigh,
You had better been there than turned loose in the sky.

. . . . . .

"In their own way of thinking, all felt and all reasoned,
Greedy aldermen judged that your flight was ill-seasoned,
That you'd better have taken a good dinner first,
Nor have pinched your poor stomach by hunger or thirst.

"In perfect indifference the beau yawned a blessing,
And feared before night your hair would want dressing;
But the ladies, all zeal, sent their wishes in air,
For a man of such spirit is ever their care.

"Attornies were puzzled how now they could sue you,
Underwriters, what premium they'd now take to do you;
While the sallow-faced Jew, of his monies so fond,
Thanked Moses he never had taken your bond."

Mr. Baldwin ascended in Lunardi's balloon, the latter being present at the start, though not taking part in the voyage. M.

[The *N.E.D.* gives nothing earlier than 1783 (*Europ. Mag.* iv. 272, "Monsieur de Montgolfier's Air Balloon"; Cowper's Letters, Sep. 29, "What is your opinion of these air balloons?").

J. Tytler ascended from the Comely Gardens, Edinburgh, Aug. 27, 1784, and "had the precedence by a few days" of Vincent Lunardi (*Encyc. Brit.* iii. 264, where Sept. 15th is given as the date of the ascent by the Italian). Tytler was "virtually the editor" of the 2nd edit. of the *Encyc. Brit.* Lunardi was secretary to Prince Caramanico, the Neapolitan ambassador.]

XXV.

I. ii. 326. **Alteration of Title-pages.**—Among the advertisements in the last *Quarterly* and *Edinburgh Reviews*, is one which replies to certain criticisms on a work. One of these criticisms was a stricture upon its title. The author states that the reviewer had a *presentation copy*, and ought to have inquired into the title under which the book was sold to the *public* before he animaverted upon the connexion between the title and the work. It seems then that, in this instance, the author furnished the *Reviews* with a title-page differing from that of the body of his impression, and thinks that he has a right to demand that the reviewers should suppose such a circumstance probable enough to make it imperative upon them to inquire what the real title was. Query, Is such a practice common? Can any of your readers produce another instance? M.

[*Animaverted*; must be a misprint. No such form is found in *N.E.D.*]

XXVI.

I. ii. 374. **"Singular"** *and* **"Unique."**—The word *singular*, originally applied to that of which there is no other, gradually came to mean extraordinary only, and "rather singular," "very singular indeed," and such like phrases, ceased to shock the ear. To supply the vacancy occasioned by this corruption, the word *unique* was introduced; which, I am horror-struck to see, is beginning to follow its predecessor. The Vauxhall bills lately declared Vauxhall to be "the most unique place of amusement in the world." Can anything be done to check this ill-fated word in its career? and, if not, what must we look to for a successor? M.

XXVII.

I. ii. 424. **Poisons.**—Our ancestors believed in the existence of poisons made so artfully that they did not operate till several years after they were administered. I should be greatly obliged by any information on this subject obtained from English books published previously to 1600. M.

XXVIII.

I. ii. 442. **Calendar of Sundays in Greek and Romish Churches.**—Where can I find good authority on the calendar of Sundays in the Greek Church, and in the Roman? As to the latter, the missals and directories only give the current year; as to the former, there is no work I know of which gives anything. M.

D. C. replied that for a calendar of both Churches "M" need seek no further than—the *Almanack de Gotha* for 1851.

XXIX.

I. iii. 60. **Merrick and Tattersall.**—Will any of your readers or correspondents be so obliging as to give the years of the *birth* of Merrick, the poet and versifier of the Psalms, and of his biographer Tattersall? The years of their deaths are given respectively 1769 and 1829; but I can nowhere find when they were born. M.

[Merrick was born in 1720, and Tattersall in 1752. Lives of both are given in the *D.N.B.*]

XXX.

I. iii. 149. **Obeism.**—As your correspondent T. H. desires "any information" on the subject of *Obeism*, in the absence of more and better, I offer my mite: that in the early part of this century it was very common among the slave-populatior in the West Indies, especially on the remoter estates—of course of African origin—not as either a "religion" or a "rite," but rather as a superstition; a power claimed by its professors, and assented to by the *patients*, of causing good or evil to, or averting it from them; which was of course always for a "consideration" of some sort, to the profit, whether honorary, pecuniary, or other, of the dispenser. It is by the pretended influence of certain spells, charms, ceremonies, amulets worn, or other such incantations, as practised with more or less diversity by the adepts, the magicians and conjurers, the "false prophets" of all ages and countries.

On this matter, a curious phenomenon to investigate would be, the process by which the untonsured neophyte is converted into the bonneted doctor; the progress and stages of his mind in the different phases of the practice; how he begins by deceiving himself, to end in deceiving others; the first uninquiring ignorance; the gradual admission of ideas,

what he is taught or left to imagine; the faith, of [? or] what is fancied to be so, the mechanical belief; then the confusion of thought from the intrusion of doubt and uncertainty; the adoption of some undefined notions; and, finally, actual unbelief; followed by designed and systematic injustice in the practice of what first was taken up in sincerity, though even this now perhaps is not unmixed with some fancy of its reality. For this must be the gradation more or less gone through in all such things, whether Obeism, Fetichism, the Evil Eye, or any sort of sorcery or witchcraft, in whatever variousness of form practised; cheats on the one hand, and dupes on the other: the *primum mobile* in every case being some shape or other of *gain* to the practitioner.

It seems, however, hardly likely that Obeism should now be "rapidly gaining ground again" there, from the greater spread of Christianity and diffusion of enlightenment and information in general since the slave-emancipation; as also from the absence of its feeding that formerly accompanied every fresh importation from the coast: as, like mists before the mounting sun, all such impostures must fade away before common sense, truth, and facts, whenever these are allowed their free influence.

The conclusion, then, would rather be, that Obeism is on the decline; only more apparent, when now seen, than formerly, from its attracting greater notice. M.

XXXI.

I. iii. 241. **Saints' Days.**—The *chorea invita* is not a very satisfactory explanation of St. Vitus's dance; and though St. Vitus is not in the Roman martyrology of our day, yet he is in the almanacs of the fifteenth century, and probably earlier. The martyr Vitus makes the 15th of June a red-letter day in the first almanac ever printed. Who was St. Vitus, and how did he give his name to the play of the features which is called his dance? Again, the day before St. Patrick is celebrated in Ireland, St. Patricius is celebrated in Auvergne. Can any identity be established? M.

XXXII.

I. iii. 243. **Southey.**—There is a *jeu d'esprit* attributed to Southey, on the expedition of Napoleon into Russia, beginning,—

> "Buonaparte must needs set out
> On a summer's excursion to Moscow,"

and ending:

> "But there's a place which he must go to,
> Where the fire is red, and the brimstone blue,
> Sacre-bleu, ventre-bleu,
> He'll find it hotter than Moscow."

I know this was printed, for I saw it when a boy. Where can it be found? M.

[The editor gave the reference: "The March to Moscow," in Southey's *Poetical Works*, p. 464, edit. 1850.]

XXXIII.

I. iii. 264. **Who was the Author of "The Modest Enquiry, &c."?**—There is an anonymous tract, entitled *A Modest Enquiry, &c.* (4to, London, 1687), on the question of St. Peter's ever having been at Rome: proving, in so far as a negative in the case can be proved, in the most logical, full, clear, and satisfactory manner, that—*He never was at Rome*; and *never was, either nominally or otherwise, Bishop*

of the Church there: and showing the grounds for the contrary assertion to be altogether baseless and untrue; being a tissue of self-contradicting forgeries and frauds, invented long subsequently to the time, evidently for the sole purpose of justifying the Papal pretensions of succession and derivation from the Apostle; as those, and all its other claims, are founded alone on that fact, and must stand or fall with it.

The inquiry is conducted throughout with evidence of great acquaintance with Scripture and much theological learning (though the writer states himself to be a layman), without the least undue pretension, and with the most perfect temperateness and impartiality. The work would seem now well worth while reprinting in a cheap and popular form.

Who was the author? M.

[The Editor replies: In Francis Peck's *Catalogue of Discourses in the Time of King James II.*, No. 226, the name of HENRY CARE is given as the author. A list of his other works may be found in Watt's *Bibliotheca*. The *D.N.B.* gives Henry Care (1646-1688) as a journalist who edited a partisan anti-Romanist journal, called *The Weekly Pacquet of Advice from Rome*, 1678-1679. It was suppressed in 1680.]

XXXIV.

I. iii. 434. **Mrs. Catherine Barton.**—Your correspondent will find all that is known in Sir David Brewster's *Life of Newton*, and will see (p. 323) that her maiden name must have been either Smith, Pilkington, or Barton itself. M.

I. viii. 258. When I answered the Query, I was not aware of what Baily states in the Supplement to Flamstead, p. 750. Rigaud ascertained for Baily that Mrs. C. B. (the title *Mistress* being given at that period to marriageable young ladies) was not the *wife*, but the *sister* of Colonel Barton. Both were the children of Hannah Smith, Newton's half-sister, and Robert Barton. Mrs. C. B. was born about 1680. M.

XXXV.

I. iii. 464. "The Borderer," in a paper maintaining against Jeffrey, "the Aristarchus of the *Edinburgh Review*," that the Goblin Page is the "keynote" to the *Lay of the Last Minstrel*, wrote: "It is amusing to notice a slight 'incuria' on the part of the poet, which I wonder has never been corrected in the later editions. Having described the nuptial ceremony of Cranstour and Margaret in the early part of the last Canto, he says in Section xxviii.:

> 'Nought of the bridal *will* I tell,
> Which *after* in short space befell,' etc., etc."

To this De Morgan replies:

Lay of the Last Minstrel.—"The Borderer," with whom, I fancy, every one will fully agree, has himself been guilty of *incuria* in charging it upon Walter Scott. The great festival at which Michael Scott marches off with the Goblin Page, was to celebrate, not the *nuptials*, but the *betrothal*, of the hero and heroine. I do not think I have read the *Lay* since I was a boy; but yet I will bet five nothings to one, that the following lines are spoken by the Lady, when she gives way, as she says, to Fate:

> "For this is your *betrothing* day,
> And all these noble lords shall stay
> And grace it with their company."

(*To be continued.*)

MATHEMATICAL NOTES.

634. [J. 2.] *Notes* 540 (vol. x., p. 43) *and* 560 (vol. x., p. 176).

The result can be simplified by combining the last two terms in { } to give

$$\binom{10}{5}x(1-x)^5\Big/(1-2x\sqrt{1-x}).$$

This can also be reached directly. The chance of "five all" is $\binom{10}{5}x^5(1-x)^5$. If this happens, let A's chance of winning be p. He wins if he gets next two games, chance x^2; if he gets one and loses the other [chance $2x(1-x)$], his chance of winning set is again p. Hence

$$p=x^2+2x(1-x)p \quad \text{and} \quad p=x^2/(1-2x\sqrt{1-x}),$$

whence the above result follows at once—remembering the x^6 outside the { }.

6th May, 1922. G. J. LIDSTONE.

635. [K¹. 1. c.] *On Note* 621 (vol. xi., p. 85).

"Morley's" Problem was set in the St. John's group of Entrance Scholarships, Dec. 1920. It appeared earlier as Qn. 16381 (E. J. Ebden), *Mathematics from the Educational Times*, N.S. vol. xv. (1909), pp. 22, 46, 110.

G. N. BATES.

636. [I. 1.] *An Introduction to Logarithms.*

I have just had the opportunity of introducing logarithms to a class of apprentices who left their primary schools three, four, and five years ago, and who are now learning their trades—linotype operator, linotype engineer, electrician, mechanic, process operator—at the *Sydney Morning Herald* and *Sydney Mail*. The proprietors of these papers are running their own school under a generous scheme of Continued Education. Every member of the staff under twenty-one attends the school, not only the apprentices but also the messengers, clerks, accountants, and the like; there were several of the latter in the group which attacked logarithms.

It is necessary also to explain that there are no examinations to work for. The school hours, in the employers' time, are devoted to anything and everything which may help to divert the workers' brain to something above his vocational training, and it may safely be said, without enlarging upon details, that the scheme is more than paying for itself as a contribution to a solution of the problems which everywhere now confront both employer and employed.

The attempt to solve a simple question on the flight of an aeroplane led to Pythagoras' theorem. (These lads had never done any formal geometry.) From this they raised the question as to whether there was not any shorter way of doing Square Roots. Again, one youth actually hit upon the possibility of using a sort of standard right-angled triangle for all other triangles of the same shape; in brief, he discovered the principle of the trigonometrical ratios, and the development of his idea has been promised for future discussion. Meanwhile, I put up on the board in a column

$$10^2=100, \qquad ?=318, \qquad 10^3=1000.$$

The class had already had some drill in Indices, but they were not quite seized with the generality of $a^m.a^n=a^{m+n}$. It did not take long to convince them that only a dimensions-hound could picture $a^3.a^2=a^5$, and, given fractional indices, even he might be disposed to accept the axiomatic rule without question.

The suggestion was soon made that 318 must be 10 to some power in between 2 and 3. Without further ado I gave

$$10^{2\cdot5024} = 318, \quad \ldots\ldots(1)$$

with an immediate warning that, if I liked, I could go further with the decimal.

Some University teachers refuse to give Taylor's Theorem until the learner can appreciate a rigid proof. Similarly, some people would not allow Logarithms to be used until the learner could make his own tables. Sure enough, one bright lad asked how the log. table was arrived at; I countered with the enquiry as to how he digested his food, and the wrangle ended with the comforting assurance all round that we did and used lots of things every day without knowing the fundamental how or why. However, whilst nobody yet knows all about the process of digestion it is possible to know all about the making of log. tables, and the matter will receive attention later.

The next step was to multiply both sides of the equality (1) by 10. Hence

$$10^{2\cdot5024} \cdot 10^{1} = 10^{3\cdot5024},$$

and so

$$10^{3\cdot5024} = 3180. \quad \ldots\ldots(2)$$

A very little drill impressed the facts, without the portentous text-book verbiage,

that multiplication or division by any integral power of 10 could not affect the decimal part of the index;
that another name for index is "logarithm";
that the logarithm consists of two parts, integral and decimal;
that all numbers with the significant figures 318 have the same decimal part for their logarithms when the said numbers are written in the "standard form," *i.e* to base 10.

The following tables were now reached by the forced application of an obvious regularity:

$10^{3} = 1000$	$10^{3\cdot5024} = 3180$
$10^{2} = 100$	$10^{2\cdot5024} = 318$
$10^{1} = 10$	$10^{1\cdot5024} = 31\cdot8$
$10^{0} = 1$	$10^{0\cdot5024} = 3\cdot18$
$10^{-1} = \cdot1$	$10^{\bar{1}\cdot5024} = \cdot318$
$10^{-2} = \cdot01$	$10^{\bar{2}\cdot5024} = \cdot0318$
$10^{-3} = \cdot001$	$10^{\bar{3}\cdot5024} = \cdot00318$

"For hundreds, 2; for hundredths, bar 2" was made the mnemonic for all determinations of the integral part of the logarithm. The usual rules given in the text-books are exceedingly unwise in their verbose repulsiveness. The putting of the negative sign above the integer was regarded as a 'cute little son of necessity, sired by a commonsense mathematician.

And so the game went on, along obvious lines. As examples of manipulation we had the following:

1. *Find* 318×4208.

$$318 \times 4208 = 10^{2\cdot5024} \times 10^{3\cdot6240} = 10^{6\cdot1264} = 1{,}338{,}000.$$

This result emphasized the approximateness of the tables. We were all satisfied that for income tax purposes we should ignore the hundreds.

2. *Square* 318.

$$318^{2} = (10^{2\cdot5024})^{2} = 10^{5\cdot0048} = 101{,}100.$$

The rule $(a^{m})^{n} = a^{mn}$ was easily recognised as a consequence of the fundamental index law.

3. *Find the square root of* 318.

Since $a^{\frac{1}{2}} . a^{\frac{1}{2}} = a$, therefore $a^{\frac{1}{2}}$ is the square root of a.

$$318^{\frac{1}{2}} = (10^{2\cdot5024})^{\frac{1}{2}} = 10^{1\cdot2512} = 17\cdot83.$$

(Check the fact that the result must be in 10's by reference to the usual way of finding square roots.)

4. Cube roots follow at once. The real value of logs. is of course felt here first by people who, up to now, say they can use ordinary arithmetical processes more quickly than they can use logs.

5. The tables supplied use only the base 10. What would 318 be to the base 3 ?

Let $318 = 3^x$. Then $10^{2\cdot5024} = 10^{0\cdot4771\,x}$; $\therefore\ x = \frac{2\cdot5024}{0\cdot4771}$.

There is no side-working.

Plenty of little points arise for settlement, but it is not necessary to note them here.

If, or when, the time comes to put the subject on the usual generalised basis, then it should not be forgotten that in showing, for example,

$$\log(ab) = \log a + \log b,$$

the method consists essentially in *operating* on the indices-forms and *translating* into logarithm-forms. Thus :

	Indices.	*Logarithms.*
Let	$a^y = N$	$\log_a N = y$
and	$a^z = M.$	$\log_a M = z$
Then	$a^y . a^z = NM,$	$\log_a NM = y+z$
i.e.	$a^{y+z} = NM$	$= \log_a N + \log_a M.$

It seems, however, bad pedagogy even with more academic classes to bother much about the "spit and polish" of the numerous rules. After all, the operation is the thing ; and the method of working by indices is not only perfectly clear but it offers an immense insurance against error, not only by making the whole working read grammatically from beginning to end, but also by eliminating the side-working, which is a constant source of trouble because it induces forgetfulness of the progress of the main problem. How often an examinee serves up a log. instead of the antilog. !

Of course, the poor teacher who has to grind for an exam. has not the delightful advantage of my freedom, nor has he often the pluck to realise that he can be equally free and yet be successful in getting results !

Sydney. F. G. Brown.

637. [E¹. 11. e.] *Note on Note* 631 (vol. xi., p. 89).

Since writing my note on a Porism of Lord Brougham, I have noticed that the conditions in the particular poristic case that I discovered signify that the three fixed circles should belong to a coaxial system. I have also found that the same set of conditions determine a poristic case for another locus, viz. that defined by the condition $(m+n) . OT_1 = m . OT_2 + n . OT_3$.

This locus is in general a circular cubic, but breaks down into a circle and a straight line, irrespective of the value of the ratio $m : n$, when the three fixed circles belong to a coaxial system.

In fact I find that the locus obtained by equating to zero any homogeneous integral function of $OT_1^{\frac{1}{2}}$, OT_2, OT_3, consists entirely of circles, when the three original circles form part of a coaxial system, and that the circles forming the locus belong also to the coaxial system. J. Brill.

638. [K¹. 2. d.] *Isogonal Conjugates.*

I. Let S, S' be isogonal conjugates for $\triangle ABC$, and let AS, AS' meet the circumcircle in L, L'. Draw AD, Sa, $S'a'$ perpendicular to BC, $S'c'$ perpendicular to AB, and let AOA', BOB' be diameters of the circumcircle.

Consider the $\triangle$s SBL, $BS'L'$.

$\angle BLS = \angle BL'S'$ in same segment.

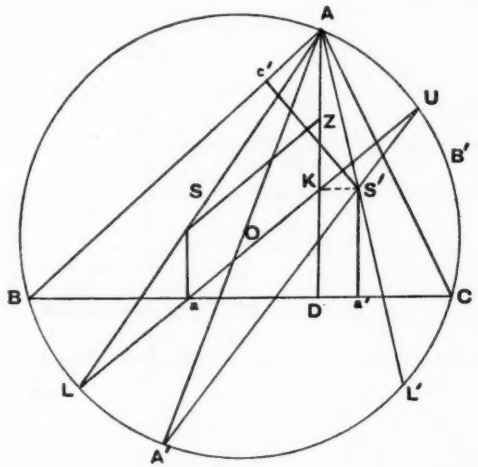
$\angle BSL = \angle SAB + \angle SBA = \angle L'BC + \angle S'BC$, since arc BL = arc CL',
$= \angle S'BL'$;

$\therefore$ the triangles are equiangular;

$$\therefore\ SB : SL = S'B : BL'.$$

But $Sa : SB = S'c' : S'B$, since right-angled $\triangle$s SaB, $S'c'B$ are equiangular;

$$\therefore\ Sa : SL = S'c' : BL' \textit{ ex aequali}.$$



Now the right-angled triangles $S'Ac'$, $BB'L'$ are similar, since

$$\angle c'AS' = \angle BB'L' \text{ in same segment};$$

$$\therefore\ S'c' : BL' = AS' : 2R,$$

so
$$Sa : SL = AS' : 2R.$$

Hence
$$2R \,.\, Sa = SL \,.\, AS'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly
$$2R \,.\, S'a' = S'L' \,.\, AS. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying
$$4R^2 \,.\, Sa \,.\, S'a' = AS \,.\, SL \,.\, AS' \,.\, S'L',$$

so
$$4R^2 \,.\, \rho_2^{\,2} = (R^2 - OS^2)(R^2 - OS'^2),$$

[R. W. Genese.]

where $2\rho_2$ is minor axis of inscribed conic.

Corollary I. Since
$$\angle LSa = \angle A'AS'$$

and
$$Sa : SL = AS' : AA',$$

the triangles SLa, $AA'S'$ are equiangular.

Corollary II. If SZ is drawn parallel to La to meet AD in Z, the triangles AZS, $AS'A'$ are equiangular, whence

$$AS \,.\, AS' = 2R \,.\, AZ.$$

Corollary III. If La meets AD at K and $A'S'$ at U, U is on the circumcircle, and A, K, S', U are concyclic, so $\angle AKS'$ is a right angle.

Corollary IV. If Ω is the mid point of SS' and Ωn is perpendicular to BC,

$$ZD = ZK + KD = Sa + S'a' = 2\Omega n,$$

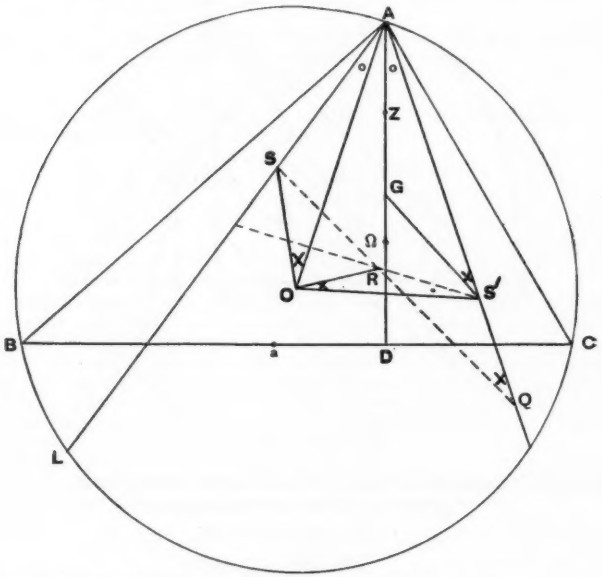
whence

$$\Omega Z = \Omega D.$$

II. On OS' and $S'A$ draw triangles ORS', $S'GA$ each equiangular to $\triangle OSA$. Then G lies on AD.

Now $\quad OS : OR = OA : OS'$ and $\angle SOR = \angle AOS'$;

$\therefore$ $\triangle$s OSR, OAS' are similar.



Hence, if SR cuts AS' in Q, $\angle OSR = \angle OAQ$, so $ASOQ$ lie on a circle;

$$\therefore \angle SQA = \angle SOA = \angle AS'G.$$

$\therefore$ $S'G$ is parallel to SR, but from similar triangles

$$\frac{SR}{AS'} = \frac{OS}{OA} = \frac{S'G}{AS'};$$

$$\therefore SR = S'G.$$

Hence $SRS'G$ is a parallelogram. $\therefore$ Ω is the middle point of RG. From similar triangles $AS \,.\, AS' = AO \,.\, AG$, but (Corollary II.)

$$AS \,.\, AS' = 2AO \,.\, AZ;$$

$\therefore$ Z is the middle point of AG, so $AR=2\Omega Z=2\Omega D$, and AR and ΩD are equally inclined to AD. Again $AO=2ND$, where N is the 9-point centre, and these lines also are equally inclined to AD;

$$\therefore \angle OAR=\angle ND\Omega,$$

so $\triangle$s OAR, $ND\Omega$ are similar, and $OR=2N\Omega$.

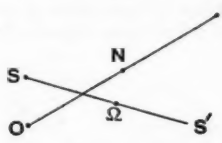
Hence $OS\,.\,OS'=OA\,.\,OR=2R\,.\,N\Omega$. .....[V. R. Aiyar.]

III. Let $2\rho_1$ be major axis, and H the orthocentre. Combining the two previous results, $R^2-(OS^2+SO'^2)+4N\Omega^2=4\rho_2^2$;

$$\therefore R^2-(2O\Omega^2+2\Omega S^2)+2O\Omega^2+2\Omega H^2-4ON^2=4\rho_1^2-4\Omega S^2,$$

so $$4\rho_1^2=R^2-OH^2+2\Omega S^2+2\Omega H^2;$$

$$\therefore 4\rho_1^2=R^2-OH^2+SH^2+S'H^2.$$



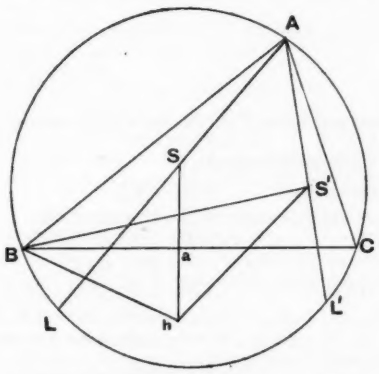
IV. Again, since $\Omega S^2=\rho_1^2-\rho_2^2$ and $OH^2-R^2=2\rho^2$, where ρ is radius of polar circle, $(\rho_1^2+\rho_2^2)+\rho^2=\Omega H^2$;

$\therefore$ the polar circle of $\triangle ABC$ cuts orthogonally the director circle of any inscribed conic.

V. Produce Sa to h making $ah=Sa$; then $S'h$ is major axis of in-conic. Since $\triangle$s BSL, $S'BL'$ are similar,

$$BS:BS'=SL:BL';$$

$$\therefore Bh:BS'=SL:LC \quad (\text{arc } BL'=\text{arc } LC).$$



Also $\angle CLS=B=\angle CBS'+\angle CBS=\angle S'Bh$;

$\therefore$ $\triangle$s BhS', LSC are similar, whence

$$S'h\,.\,SL=Bh\,.\,SC;$$

$$\therefore 2\rho_1=\frac{SB\,.\,SC}{SL}. \quad \text{.....[R. F. Davis.]}$$

VI. *Corollary.* $2\rho_1 = \dfrac{S'B \cdot S'C}{S'L'}$, similarly;

$$\therefore\ 4\rho_1^2 = \frac{SB \cdot SC \cdot S'B \cdot S'C}{SL \cdot S'L'} = \frac{SA \cdot SB \cdot SC \cdot S'A \cdot S'B \cdot S'C}{SA \cdot SL \cdot S'A \cdot S'L'};$$

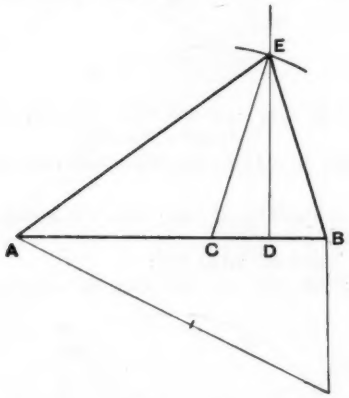
$$\therefore\ 16\rho_1^2\rho_2^2R^2 = SA \cdot SB \cdot SC \cdot S'A \cdot S'B \cdot S'C.$$

VII. Z is clearly on the tangent to the conic which is parallel to BC, and if LP is perpendicular to BC it can readily be proved that $LP \cdot AZ = \rho_2^2$.

E. P. LEWIS.

639. [K¹. 3. a.] *To describe an isosceles triangle having each of the base angles double of the vertical angle.*

Construction. Draw any line AB and divide it at C so that $AB \cdot BC = AC^2$. Bisect BC at D and draw DE at right angles to BC.



With C as centre and radius CA describe a circle cutting DE at E.
Join BE.
Then ABE is the required triangle.

Proof. By construction $CA = CE$;

$\therefore$ exterior angle $BCE = 2^{cs}$ angle CAE.

Again $CE = BE$; $\therefore\ \angle CBE = \angle BCE$

$= 2^{cs}\ \angle BAE$.

From triangle AEB, $AE^2 = AB^2 + BE^2 - 2AB \cdot BD$

$= AB^2 + AC^2 - AB \cdot BC$

$= AB^2$, since $AB \cdot BC = AC^2$;

$\therefore\ AE = AB$.

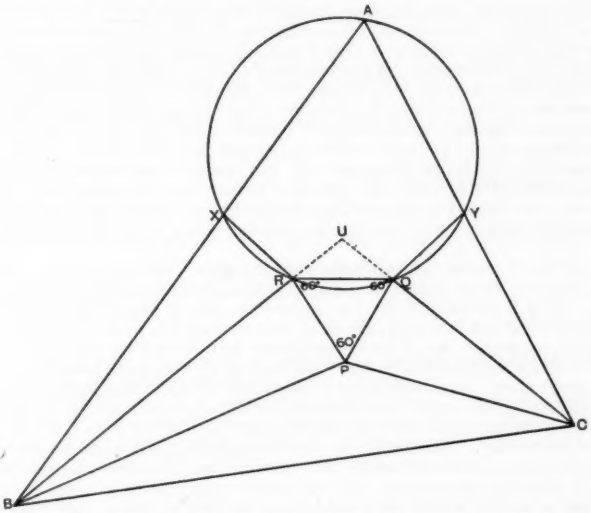
Hence the triangle ABE is isosceles, having each of the angles ABE, AEB double of the angle BAE.

This construction and proof make no use of circle properties other than the equality of radii. They are based on the method given in *Exercises in Paper Folding*, by T. S. Row, a copy of which is in the Library of the Association.

W. E. PATERSON.

640. [K¹. 1. c.] *Proof of "Morley's Theorem"* (by Euclid, Bk. III.).

Trisect the angles B, C: let the trisectors adjacent to BC meet in P; make $B\hat{P}R$, $C\hat{P}Q$ respectively $60+C/3$ and $60+B/3$: then, if PR, PQ meet the other trisectors BR, CQ in R, Q respectively, $B\hat{R}P=60+A/3=C\hat{Q}P$ and $Q\hat{P}R=60°$.



But, since PQ, PR are equally inclined to the sides of the triangle BUC, of which P is the incentre, it follows that $PQ=PR$.

Hence $\triangle PQR$ is equilateral.

Make $BX=BP$ and $CY=CP$; then $XR=RQ=QY$.

Also $$\angle XRQ=180°-2A/3=\angle RQY.$$

Therefore a circle can be described through X, R, Q, Y; and the arcs XR, RQ, QY will subtend angles $A/3$ at the rest of the circumference, *i.e.* $XRQY$ subtends an angle A. Therefore A is on this circle and AR, AQ trisect angle A.

J. M. CHILD.

641. [V. a. ζ.] *Reduction to Absurdity.*

The method of reduction to absurdity in Geometry is perhaps just a little out of fashion at present: anyhow there is a distinct tendency to replace proofs which employ it by direct ones.

Still the method is after all the fundamental method in Geometry for dealing with converses of theorems already proved: if there is any doubt as to the truth of a converse theorem, the first line of enquiry is likely to be, "Can it be proved by reduction to absurdity?"

This being so, it is essential that the method should be learnt by students of Geometry, and it becomes of some interest to determine at what point it should first be introduced.

In the Euclids of twenty years ago it came in first at I. 6 to prove a triangle isosceles; was used in the first proposition on parallels; in the version "proof by exhaustion" to prove the greater side opposite the greater angle; for

areas in proving equal triangles on the same base to lie between the same parallels; but I do not think it was used to prove the converses of theorems on angles in a circle except perhaps in an appendix.

Nowadays much of this is changed. In the first place the old phrase "which is absurd" more often than not becomes "which is contrary to hypothesis" or "which is impossible." This may be regarded as an unimportant matter of taste (in fact probably an improvement except for the hurried writer). But even with this variation the method is much less used.

For the isosceles triangle a direct proof is usually substituted either by superimposing the triangle reversed on itself or more likely by using the "two angles and side" case of congruence which has been worked in earlier. For the parallels, as likely as not, for Euclid's simple proof, a weird and wonderful version not using the exterior angle but twisting the left half of the crooked and sideways *H* and placing it on the right half, is substituted. In this it is true that the pupil is asked to agree that two straight lines cannot meet twice, but he is not likely to learn the method of reduction to absurdity from this.

In any case these two propositions are probably coaxed into the beginner by intuitive methods and not tackled seriously till a much later revision, if then.

As to the "greater side opposite greater angle," this may have a direct proof, if the textbook thinks two sides of a triangle obviously greater than the third. Even if it has not the two alternative wrong hypotheses complicate matters, so that this is scarcely the ideal place to learn this method, and anyhow these inequalities are apt to be side-tracked; in a book I am getting to know very well they are placed after the extensions of Pythagoras' theorem.

To prove that "equal triangles on same side of same base lie between the same parallels," it is now not unusual to argue (arithmetically) that their altitudes must be equal, and that as these are parallel the lines to be proved parallel are so also. This proof may be direct, but I think most teachers will agree with me that it is far harder than the old one. However, another possibility is that this proposition may be merely regarded as a corollary to its converse, a method which the pupil is only too glad to apply to all converses whether true or false.

There remains the proof that $ABCD$ is cyclic if $\angle$s ABD, ACD are equal. If this proof is found difficult by a pupil it is often because here for the first time he is striving to master the method of reduction to absurdity.

The question I wish to ask is, where is the right place to bring this method in first? Of the propositions mentioned the one which seems to me to have fewest difficulties to distract attention from the new method is that about parallels, or if that is to be postponed (to the Euclidean Kalends) the one about the equal triangles.

Or it might be well to prove that the join of the mid points of the sides of a triangle is parallel to the base by drawing a parallel from one mid point and insisting that it is absurd to suppose a line to have two mid points.

If this is done, or if the method is first used in the converses dealing with angles in a circle, we have the curious state of affairs that one of Euclid's most characteristic methods of proof is introduced in a proposition not included in those of Euclid.

C. O. TUCKEY.

642. [K¹. 13. c.] *Expression for Volume of a Tetrahedron.*

Find the volume of a tetrahedron with one vertex at the origin; that for any tetrahedron follows at once.

Let $OPQR$ be a tetrahedron with a vertex at origin O. If the plane of PQR be produced to meet the co-ordinate axes in A, B, C, the volumes of $OPQR$ and $OABC$ are as the areas of the triangles PQR and ABC, which are as the areas of their projections on the plane of yz; that is, as

$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ is to $OB \,.\, OC$. Thus 6 vol. $OPQR = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \times OA$.

But writing equation to plane through P, Q, R, and putting $y=0$, $z=0$, we get

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \times OA = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \equiv (x_1 y_2 z_3), \text{ suppose.}$$

Now, since any tetrahedron $PQRS$ is equal to the algebraic sum of the four tetrahedra whose common vertex is O, and bases the faces of $PQRS$, we have

$$6 \text{ vol. } PQRS = (x_2 y_3 z_4) - (x_1 y_3 z_4) + (x_1 y_2 z_4) - (x_1 y_2 z_3)$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}, \text{ the required expression.}$$

When the axes are oblique, we multiply this expression by $2n$, where

$$4n^2 = 1 - \Sigma \cos^2 \alpha + 2\Pi \cos \alpha,$$

α, β, γ being the angles between the co-ordinate axes.

Univ. Coll., Cork. A. H. ANGLIN.

643. [K¹. 9. b.] The following construction in Practical Solid seems worth noting. It has useful applications in Crystallography.

Let $OAMBLCHK$ be a rectangular parallelopiped. It is required to find the point at which the normal through O to the plane ABC meets the face $CHKL$. Draw perprs. from O to CA, CB meeting CH, CL in D, E. Complete the rectangle $DCEF$. Then OF is perpr. to pl. ABC.

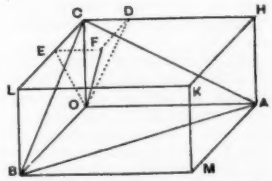


FIG. 1.—The parallelopiped in oblique parallel perspective.

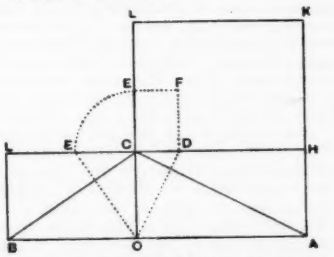


FIG. 2.—The planes $COBL$, $CLKH$ rabatted into the plane of the paper, that of $OAHC$, round CO, CH.

If OC is the axial unit along OZ, and if OA, OB along OX, OY contain m and n units respectively, then F is the gnomonic projection of the pole of pl. ABC $\left(\frac{x}{m}+\frac{y}{n}+z=1\right)$ with respect to the unit sphere with centre O. In this case the construction admits of simplification.

EDWARD M. LANGLEY.

644. [K¹. 1. c.] *A Problem.*

ABC is an isosceles triangle. $B=C=80°$. CF at 30° to AC cuts AB in F. BE at 20° to AB cuts AC in E. Prove $B\hat{E}F=30°$. E. M. LANGLEY.

REVIEWS.

The Rudiments of Relativity. By J. P. DALTON. Pp. vi+105. 5s. 1921. (Council of Education, Witwatersrand, Johannesburg.)

Le Principe de Relativité et la Théorie de la Gravitation. By J. BECQUEREL. Pp. ix+342. 25 francs. (English price, 12s. 6d.) 1922. (Gauthier-Villars.)

Space—Time—Matter. By H. WEYL. Translated by H. L. BROSE. Pp. xi+330. 18s. 1922. (Methuen.)

A Criticism of Einstein and his Problem. By W. H. V. READE. Pp. 126. 4s. 6d. 1922. (Blackwell, Oxford.)

La Notion d'Espace. By D. NYS. Pp. 446. 15s. 1922. (Oxford University Press.)

Of these books Weyl's is by far the most important, but it will be convenient to discuss the more elementary ones first.

The study of relativity may be divided into four stages. The first of these is purely descriptive. There are many books which endeavour to give a general idea of Einstein's theories without mathematics. They describe the Michelson-Morley experiment, and state the two postulates of the restricted theory and the deductions from them concerning measurements of length and time and their dependence upon the relative motion of the observer and the object observed. In the generalised theory they state the three well-known deductions concerning the perihelion of Mercury, the deflection of light by gravitation, and the shift of spectral lines. It is possible to make clear dogmatic statements concerning all these matters. It is when they try to give a verbal explanation of how these deductions are obtained that they become confusing.

Prof. Dalton's is one of the best of the descriptive books. He is well aware of the limitations of a non-mathematical treatment and candidly says of Einstein's law of gravitation :—" I cannot tell you what that law is, for it is not expressible in words." The account of the restricted theory points out that it is founded not on the Michelson-Morley experiment alone but upon the apparent clash between it and the phenomena of aberration and Fizeau's experiment. The ether may be carried along by the earth in its motion, or it may be wholly or partly stagnant. It was the difficulty of accepting all three of these conclusions that gave rise to relativity.

The second stage, the full comprehension of the restricted theory, is well within the reach of all with a little mathematical knowledge. The most advanced part is merely the change of variables in partial differentiation. Some knowledge of Maxwell's electromagnetic equations is necessary, but almost any treatise on electricity will supply this. These equations really form the foundation of the whole theory. The Lorentz transformation owes its importance to the fact that it leaves the form of Maxwell's equations unchanged, provided suitable relations be assumed connecting the electric and magnetic fields in two systems moving relatively to each other with uniform velocity. These relations give us at once the fields due to a moving electron or magnet. In fact the electrodynamics of moving bodies is much simpler on the relativity theory than on the older ones. But now comes the difficulty of dealing with dynamics of material bodies. Newton's Laws of Motion lead to equations which certainly are altered by the Lorentz transformation. Electrodynamics and Newtonian dynamics clash, but as it is the electrodynamical experiments that admit of the higher degree of precision, we modify Newtonian dynamics to conform with electrodynamics and the Lorentz transformation. The mass M of a body referred to a set of axes relative to which its velocity is v is taken as $m/(1-v^2/c^2)^{\frac{1}{2}}$, where m is constant and c is the velocity of light in free space. Momentum is defined as the product of mass and velocity, and force as the rate of change of momentum. But as the mass is variable, force is no longer the product of mass and accelera-

tion, except at the instant when the velocity is zero. Kinetic energy is defined as $Mc^2 - mc^2$, reducing approximately to $\frac{1}{2}mv^2$ for small values of v.

The best exposition of these results will be found in Einstein's original paper * of 1905. Of course some of the mathematics is left to the reader. Those who doubt their ability to fill in the missing steps may consult M. Becquerel's book, which will be found very helpful. It does not aim at originality, but reproduces the actual words of Einstein and Eddington as far as possible, with a simplified treatment of the more difficult parts. The range is a wide one, including Einstein's views on the ether and on the finiteness of space, and Weyl's and Eddington's extensions of Einstein's ideas. The experimental verifications are emphasised. The dependence of mass upon velocity is confirmed by Bucherer's experiments on β-rays, Guye and Lavanchy's experiments on cathode rays, and numerous observations on the spectral lines of hydrogen. Those who reject the theory of relativity must find alternative explanations of all these.

The third stage, dealing with the generalised theory, is much more difficult, for it cannot be understood without the *tensor calculus*. A *tensor* may be regarded as a generalised vector. (Some confusion has been caused by the unfortunate fact that the word tensor is used in Quaternions in another sense, namely as the numerical magnitude of a vector.) A displacement is a tensor of the first rank, while the moment of a force is one of the second rank. In ordinary mechanics they are classed together as vectors. This is only possible because space is assumed three dimensional, and two, although not equal to one, is equal to the difference of three and one. In four dimensions it is impossible to consider a tensor of the second rank, which has six † components, as equivalent to a tensor of the first rank, which has only four. These distinctions and others, such as that between a covariant and contravariant tensor, are of great importance in relativity. The essential property of a tensor is that if all its components vanish in one system of co-ordinates, they also vanish in any other. Einstein assumes that any law of nature can be expressed by equating to zero all the components of a suitable tensor. Now Newton's law of gravitation does not conform to this condition, hence Einstein replaces it by another, the vanishing of the *contracted Riemann-Christoffel tensor*. The manner in which the new gravitational theory is developed involves several assumptions and a great deal of calculation. Perhaps the least difficult form of the final result is that given by Eddington: *Every particle and light-pulse moves so that the integral of ds between two points on its track is stationary, where*

$$ds^2 = -(1-2m/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2 + (1-2m/r)dt^2,$$

in appropriate polar coordinates. This may be regarded as definitely established by observation, to a high order of accuracy. The theory of the spectral shift depends on other and less certain considerations. Although the matter is still doubtful, the latest experiments are much more favourable than the earlier ones, which are now believed to be inaccurate.

The fourth stage is that of the study of Weyl's great work. Einstein has described it as "a comprehensive and excellent text-book, to be warmly recommended to mathematicians and physicists interested in the theory of relativity." The publishers naturally give prominence to this statement. They also speak of "enabling the physicist of average attainments to grasp all the mathematical details of Einstein's theory." As a matter of fact the physicist of average attainments will probably find the book quite beyond him. The numerous new ideas introduced make the treatment appear unfamiliar. Some of the later paragraphs seem to disagree with earlier ones. The reader is expected to follow complicated mathematical arguments from which many of the steps have been omitted. On p. 137 we read with dismay

* For English translations of this and other original papers see:
The Principle of Relativity. Translated by M. N. Saha and S. N. Bose. 10s. 1920. (Calcutta University Press.)
or *The Principle of Relativity.* Translated by G. B. Jeffery and W. Perrett. 10s. 6d. 1922. (Methuen.)

† This refers only to a tensor which like the moment of a force, is *skew-symmetrical*; the general four-dimensional tensor of the second rank has sixteen ($=4^2$) components.

that "whoever sets out in quest of this goal must possess a perfect mathematical equipment from the outset." Finally, the translator in some places has completely altered the sense. For instance, on p. 81 *Euclidean* is put instead of *Non-Euclidean*, on p. 87 *are not* instead of *are*, and on p. 162 *irrotational* instead of *solenoidal*, while on p. 18 a muddle in the references makes an important passage unintelligible. However, those who know a little German can easily check doubtful passages by comparison with the original, which can be obtained from English booksellers for about 7s. 6d.

The book is divided into four long chapters, of which the first and second deal with geometry. *Affine geometry*, the geometry of those properties which are unaltered by an *affine* (that is a *linear*) *transformation*, is considered in great detail. *Metrical geometry* is then based on those particular affine transformations which leave a certain quadratic form unaltered. For ordinary two-dimensional Euclidean geometry this *metrical groundform* is x^2+y^2, which is unaltered by a change of rectangular axes with the same origin. By taking other groundforms Non-Euclidean geometries may arise. The idea of affine transformation can be extended to infinitesimal (but not to finite) displacements on curved surfaces. These surfaces also possess metrical groundforms, in which, however, the coefficients are variables and not constants. Angles, lengths of curves, and the shape and size of given regions on a surface depend only upon the metrical groundform and not upon the particular system of co-ordinates used. These properties are dealt with by the aid of the tensor calculus. When the theory of relativity is introduced it is expressed in terms of the geometrical conceptions that take up nearly half of the book.

The treatment of the generalised theory is based upon an extension of Hamilton's method of equating to zero the variation of an integral of a function. In ordinary dynamics we deal with the integral with respect to the time of the kinetic energy, or of the difference between the kinetic and potential energies. Some care must be taken in restricting the nature of the variations taken, or wrong results may be obtained. In relativity we *assume* the truth of a similar principle dealing with a fourfold integral involving the four-dimensional element of space-time and a function concerning whose nature various hypotheses may be made. Very little is said about restricting the variations, and this leaves the discussion rather unconvincing. At any rate the method is invaluable for suggesting new results.

Weyl's extension of Einstein's theory supposes a rod to be altered in length by passing near an electric current. We thus get a linear form whose coefficients relate to the electromagnetic field as well as the quadratic form determined by the gravitational. In Einstein's theory gravitation alone determines the nature of space, which is then available for electromagnetic phenomena as for lodgers in a house over whose arrangements they have no control. But in Weyl's theory electromagnetism and gravitation both have a share in determining these arrangements. Weyl adds the law of conservation of electricity to those concerning energy and momentum, accounts for Maxwell's equations, and obtains from the very beginning a *cosmological term* which is very nearly the same as that which Einstein introduces into his work at a late stage, in his views concerning the finiteness of space. However, Weyl's conclusions are not fully accepted by Einstein or Eddington, and there is no experimental evidence. Lack of space forbids us to deal more fully with this subject or to explain numerous other new ideas, such as *tensor-densities*, *calibration-invariance*, *gauge-systems*, *persistence*, and *adjustment*. This wonderful book is indispensable to all serious students.

We now turn to two books unfavourable to relativity.

Mr. Reade has been confused by the analogies used in popular explanations of the Michelson-Morley experiment. But instead of discarding analogies and turning to the full scientific discussion, he has used other analogies which, he believes, overthrow the whole theory. If he can believe this he can believe anything!

Prof. Nys gives a careful account of views that have been held on the nature of space by philosophers and others from Aristotle onwards. He recognises that the views of modern scientists must be included. Unfortunately we cannot feel sure that he really understands them. For instance, after stating the usual mathematical ideas about force he adds (p. 244): "Cette manière

de considérer la force, irréprochable peut-être en mathématiques, est cependant inadmissable en métaphysique et même en physique . . . le mouvement, même uniforme et rectiligne, est toujours, et à chaque instant, un changement réel. Or, tout changement réel démande une cause en activité." Einstein's theory takes up less than one-sixteenth of the book, and it is dismissed (p. 321) as "une conception purement mathématique, . . . mais qu'il serait au moins téméraire de regarder comme un décalque de la réalité." It is no doubt very difficult for philosophers whose training has been mainly literary to master mathematical and scientific conceptions, but until they do so their views on the nature of space are not likely to have much value. H. T. H. PIAGGIO.

CORRESPONDENCE.

TO THE EDITOR OF THE *Mathematical Gazette*.

Sir,—May I make an appeal through you to the authors on whom we depend for text-books and collections of examples? Can they be persuaded to exclude that worst of educational snares, the question to which a right answer results from using a totally wicked method? Here is a case in point:

Solve $$\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}.$$

"Clear of fractions," says the boy, and multiplies one side by ac and the other by bd;

$$\therefore\ bcx-ad=ad-bcx;$$

$$\therefore\ x=\frac{ad}{bc}.$$

The tragedy of this answer is that it is right, and its rightness confirms the boy in his belief that multiplying one side of an equation by ac and the other by bd is a good practical method, though schoolmasters may urge against it their purely academic objections.

Again consider this:

Simplify $$\frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}.$$

"Take the L.C.M." says the boy. ("Taking" the L.C.M. is a wonderful charm; what you do with it when "taken" matters nothing.)

$$\frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}=x(y-z)+y(z-x)+z(x-y)$$
$$=0.\quad \textit{Ans.}\quad \text{(Right.)}$$

In vain I may (and do) tear out handfuls of my hair and protest with purple passion that he has no right to multiply the thing by xyz. He knows better than that: "Got the answer right, but the old man wouldn't give me any marks because I didn't do it his beastly way." And deeper down in him, not formulated but very real and potent, is the logical inference, "This Mathematics is as arbitrary stuff as Latin Verbs. The masters make the rules; that's what they are for: but they have no more to do with practical common sense than the Mathematics they teach."—Yours faithfully,

Eton College, 18*th June*, 1922. W. HOPE-JONES.

PERSONAL NOTES.

The Governors of the Central Foundation Schools of London have appointed as Headmaster of their Boys' School, Cowper Street, E.C., Mr. N. M. Gibbins, M.A.(Camb.), in place of Mr. H. G. Abel, who is now Headmaster of St. Olave's Grammar School.

MATHEMATICAL ASSOCIATION.

PUBLIC SCHOOLS COMMITTEE.

Report on Mathematics in Entrance Scholarships to Public Schools.

There is great diversity of practice in the range of work covered by these papers; and it has been brought to the notice of the Mathematical Association that this diversity causes some difficulty in preparatory schools.

Some schools confine their questions to the matter contained in the syllabus issued by the Joint Committee of the Headmasters Conference and the Association of Preparatory Schools (reissued in 1916). This covers the syllabus for Elementary Mathematics in the School Certificate Examination of the O. & C. Joint Board, and contains in addition: fractional and negative indices, use of logarithms, variation, and simultaneous quadratics.

It also includes numerical Trigonometry, to which some reference is made in the School Certificate syllabus.

Other schools go far beyond this syllabus and examine in symbolical Trigonometry, Mechanics, and Conic Sections.

Clearly it would be a great convenience to preparatory schools if public schools could announce a common policy in this matter.

Furthermore, there is reason to fear that advanced mathematical papers in scholarship examinations tend to encourage specialisation in preparatory schools. If a boy has the ability to go beyond the elementary syllabus without disturbing the general balance of his work, his preparatory school master will no doubt allow him to do so; but it is very different when a powerful stimulus is applied from outside.

The question is whether public schools, by restricting their papers to the H.M.C. Syllabus, would handicap themselves in the selection of scholars. Obviously they would not diminish the *field* of selection.

As to the process of selection, Winchester has for many years restricted the range of work to something corresponding to this H.M.C. Syllabus, setting subtle questions on elementary work, and it is to be presumed that it has been found possible to discover mathematical ability by means of such papers.

Accordingly the Public Schools Committee of the Mathematical Association submit to you that entrance scholarship papers in Mathematics be restricted to the subject matter of the H.M.C. Syllabus, together with elementary Trigonometry not involving complicated identities; and, further, that the relief granted to preparatory schools by the adoption of such a policy will be sensibly augmented by the public statement thereof.

July, 1922.

Signed by: W. F. Bushell, N. J. Chignall, C. Godfrey, H. K. Marsden, R. Nettell, A. Robson, A. W. Siddons, C. J. A. Trimble, C. O. Tuckey, C. J. B. Westcott, R. M. Wright.

The following members of the Committee did not sign the report: G. W. Brewster, R. C. Fawdry, F. W. Dobbs.

YORKSHIRE BRANCH.

A meeting of the Yorkshire Branch of the Mathematical Association was held at Hymers College, Hull. Several new members were elected. A paper was read by the headmaster of Hymers College on "Mathematics and Morals," in which he showed how, apart from its utilitarian value, the subject could, in the hands of a good teacher, be made to suggest admiration for the wonderful, the beautiful, and the orderly, in short, the lesson of reverence in the truest sense of that word. The paper was illustrated by work done by pupils in

his own classes, which was most instructive and suggestive to the teachers present. Mr. H. G. Forder, M.A., senior mathematical master at the college, gave a paper on "The Infinite in Mathematics." The Branch is largely indebted to the hospitality of the headmaster for an interesting and enjoyable meeting.

LONDON BRANCH.

PROGRAMME FOR 1922-23.

1922.

Oct. 14th—(Saturday)—Meeting at The London Day Training College, at 3 p.m. Paper on "The Final Stage in the Teaching of School Geometry," by Professor T. P. NUNN, M.A., D.Sc., London Day Training College.

Nov. 18th—(Saturday)—Meeting at The London Day Training College, at 3 p.m. Discussion on "The Use of Symmetry in Geometrical Teaching," to be opened by Professor C. GODFREY, M.V.O., M.A., Royal Naval College, Greenwich.

Dec. 2nd—(Saturday)—Meeting at The London Day Training College, at 3 p.m. Paper on "Knowledge, Mathematic and Mythologic," by BENCHARA BRANFORD, M.A.

1923.

Feb. 3rd—(Saturday)—Annual Meeting at The London Day Training College, at 3 p.m. Presidential Address by Sir GEORGE GREENHILL, M.A., F.R.S.

Feb. 24th—(Saturday)—Meeting at The London Day Training College, at 3 p.m. Paper on "The Teaching of Algebra," by Miss K. REEVE, Fulham Secondary School.

Mar. 24th—(Wednesday)—An Informal Dinner at 7 p.m., to be followed by Papers on "The Teaching of Mechanics," by Prof. W. M. ROBERTS, M.A., and G. GOODWILL, B.Sc., at The London Day Training College.

Obituary.

T. G. CREAK, M.A.

WE greatly regret to record the death, in his sixty-second year, of Thomas Gasquoine Creak, honorary secretary of the North Wales Branch of the Mathematical Association.

Having graduated, at Emmanuel College, Cambridge, as 27th Wrangler in 1883, Creak joined the staff of the Newcastle-on-Tyne Grammar School. Reasons of health compelled him to resign this appointment before reaching the age of thirty, and for many years he led an invalid's life near Manchester. His health improved during the last fifteen years, and since 1903 he lived near Llanberis at the foot of Snowdon.

Throughout his life Creak maintained his interest in mathematics. Several workers found in him a most painstaking and willing collaborator. He gave useful assistance to Sir George Greenhill in troublesome algebra which arises in the division of the period in elliptic functions, to Prof. G. H. Bryan in the evaluation of the coefficients in some Fourier expansions, to the late Dr. G. B. Mathews in calculating polynomials which occur in lemniscate-function theory and to the present writer in the complex multiplication of elliptic functions.

The greater part of Creak's mathematical work, however, was done in association with Lt.-Col. Cunningham. He obtained, by direct calculation, the n solutions of the congruence

$$x^n - 1 \equiv 0 \pmod{p},$$

for a large range of values of n and p (p being a prime number and n a factor of $p-1$). From these tables, some of which are already in print, the prime factors (within the limits of the table) of $a^n - 1$ can be taken by inspection.

Another table, prepared jointly by Lt.-Col. Cunningham and Messrs. H. J. Woodall and T. G. Creak, gives the least *primitive root* of the congruence

$$x^{p-1} - 1 \equiv 0 \pmod{p}$$

for the 2800 odd primes less than 25410. The primitive roots are values of x such that the remainders, when $x, x^2, \ldots x^{p-1}$ are divided by p, are all different. When $p=7$, $x=2$, the remainders in order are 2, 4, 1, 2, 4, 1, so that 2 is not a primitive root of $x^6 - 1 \equiv 0 \pmod 7$. But when $x=3$ they are in order

$$3,\ 2,\ 6,\ 4,\ 5,\ 1,$$

and 3 is the least primitive root for $p=7$.

Creak was chosen to be secretary of the North Wales Branch of the Association (the first of the local branches) when it was founded in 1907, and retained this office until his death. The members of the branch are scattered over a wide area and much of the success attained in getting them together for the meetings has been due to Creak's energy. On more than one occasion it was the privilege of members to be entertained by his mother, Mrs. Creak, at Llanberis.

With his improved health of late years he made himself thoroughly familiar with the Snowdon district and frequently acted as guide to friends desiring his services. He ascended Snowdon on foot more than 250 times, and was also at home among the Glyders, the Carnedds and other mountains in the range.

W. E. H. Berwick.

THE LIBRARY.

The Library has now been removed to 29 Gordon Square, London, W.C. 1, and Mr. W. E. Paterson has taken over the duties of Honorary Librarian.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Regulations for the Use of the Library by Members.

1. Any member of the Association is entitled to borrow books from the Library (except those marked in the catalogue with an asterisk).
2. Not more than three volumes at a time may be borrowed, and any book borrowed must be returned within one calendar month.
3. The borrower must pay carriage both ways, and will be held responsible for any loss or damage.
4. Requests for the loan of books, or for permission to consult the books in the shelves, must be made to Mr. G. D. Dunkerley, at 29 Gordon Square, London, W.C. 1.

Scarce Back Numbers.

Reserves are kept of A.I.G.T. Reports and Gazettes, and, from time to time, orders come for sets of these. We are now unable to fulfil such orders for want of certain back numbers, which the Librarian will be glad to buy from any member who can spare them, or to exchange other back numbers for them :

Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

October, 1922.

The Rhythm of Education. By A. N. WHITEHEAD. Pp. 30. 1s. 1922. (Christophers.)

A General Text-Book of Elementary Algebra. Part I. With Answers. By E. H. CHAPMAN. Pp. 143+xxii. 3s. net. 1922. (Blackie.)

Newton's Discovery of Gravitation. By F. CAJORI. Pp. 232-238. Reprint from *Univ. of California Chronicle.* April. 1922.

A Treatise on Bessel Functions and their Application to Physics. A. GRAY and G. B. MATHEWS. Second edition prepared by A. GRAY and T. M. MACROBERT. Pp. xiv+327. 36s. net. 1922. (Macmillan.)

First Course in the Theory of Equations. By L. E. DICKSON. Pp. vi+168. 8s. 6d. net. 1922. (Wiley, Chapman & Hall.)

Mathematics as a Liberal Education. An Inaugural Address, Univ. of St. Andrews, Feb. 2, 1922. By Professor H. W. TURNBULL.

Deux grands historiens des mathématiques, M. Cantor et H. G. Zeuthen. By GINO LORIA. Pp. 265-278. Reprint from *Scientia.* April 1922.

Operazioni sulle grandezze. By G. PEANO. Pp. 20. Reprint from *Atti della Reale Accad. delle Sci. di Torino.* March 1922.

Plane Geometry. By L. B. BENNY. Pp. vi+336. 10s. 6d. net. 1922. (Blackie.)

Journal of the Mathematical Association of Japan for Secondary Education. Feb. 1922.

A Curious Mathematical Title-Page. By F. CAJORI. Reprint from *The Scientific Monthly,* March 1922. Pp. 293-295.

An Introduction to Mathematical Analysis. By F. L. GRIFFIN. Pp. viii+512 +xxii. 10s. 6d. net. 1922. (Harrap.)

Leçons d'Analyse Fonctionelle. By PAUL LÉVY. Pp. 440. 35 fr. 1922. (Gauthier-Villars.)

The Definition of an Envelope. By E. H. NEVILLE. Pp. 97-107. Reprint from *Proc. Cam. Phil. Soc.* xxi. Part. 2.

La Notion de l'Espace. By D. NYS. Pp. 446. 15s. net. 1922. (Oxford University Press.)

Forms whereon to write the Numerical Calculations described in Weather Prediction by Numerical Process. By L. F. RICHARDSON. 2s. 1922. (Camb. Univ. Press.)

Frequency Arrays, illustrating the Use of Logical Symbols in the Study of Statistical and other Distributions. By H. E. SOPER. Pp. 48. 3s. 6d. net. 1922. (Cambridge Univ. Press.)

Machine Shop Mathematics. By G. WENTWORTH, D. E. SMITH and H. D. HARPER. Pp. iv+162. 5s. 6d. net. 1922. (Ginn.)

Fundamentals of Practical Mathematics. By G. WENTWORTH, D. E. SMITH and H. D. HARPER. Pp. vi+202. 5s. 6d. net. 1922. (Ginn.)

Analytic Geometry. By L. P. SICELOFF, G. WENTWORTH and D. E. SMITH. Pp. vi+290. 11s. 6d. net. 1922. (Ginn.)

Academia pro Interlingua. [Prof. Peano, Cavoretto, Torino.] Sept. 10, 1922. N. 3.

Bell's Mathematical Tables, with a Collection of Formulae, Definitions and Theorems. By L. SILBERSTEIN, Ph.D. Pp. xi+250. 16s. net. 1922. (Bell.)

Dalton Plan Assignments. II. *Mathematics and Science.* By R. BASSETT. Pp. 64. 2s. net. 1922. (Bell.)

Examples in Practical Mathematics. Part I. By A. DAKIN. Pp. viii+143+xxi. 2s. 6d. 1922. (Bell.)

Sexagesimal Fractions among the Babylonians. By F. CAJORI. Pp. 8–10. (Reprint from *American Math. Monthly.* Jan. 1922.)

American Journal of Mathematics. (Johns Hopkins Press, Baltimore.)

Oct. 1921.

On Some Properties of General Manifolds relating to Einstein's Theory of Gravitation. Pp. 213-216. J. A. SCHOUTEN and D. J. STRUIK. *Geometrical Theorems on Einstein's Cosmological Equations.* Pp. 217-221. E. KASNER. *Fermat and Hessian Points.* Pp. 222-225. C. M. SPARROW. *Infinite Systems of Differential Equations.* Pp. 226-231. W. L. HART. *Boundary Value and Expansion Problems: Formulation of Various Transcendental Problems.* Pp. 232-270. R. D. CARMICHAEL. *Reciprocity in a Problem of Relative Maxima and Minima.* Pp. 271-290. J. K. WHITTEMORE.

The American Mathematical Monthly. (Lancaster, Pa.)

Nov.-Dec. 1921.

Graphical Solutions of the Quadratic, Cubic, and Biquadratic Equations. Pp. 415-423. T. R. RUNNING. 1321, 1471, 1521, 1571, 1621, 1671, 1721, 1771, 1821. Pp. 423-427. R. C. ARCHIBALD. *Some Arithmetic Operations with Transfinite Ordinates.* Pp. 427-430. A. A. BENNETT. *Among my Autographs.* [Maupertius and Frederick the Great; Deparcieux and a Paradoxer; Gassendi; Bouillaud and Helvetius.] Pp. 430-435. D. E. SMITH. *The Shortest Circular Path from a Point to a Line.* Pp. 437-439. A. DRESDEN. *Geometric Proofs of* $(a+b)/(a-b)=\tan\frac{1}{2}(A+B)/\tan\frac{1}{2}(A-B)$. Pp. 440-443. H. C. BRADLEY, Y. YAMANOUTI, W. Y. LOVETT, with notes by R. C. ARCHIBALD. *Some Formulae of Elementary Trigonometry.* Pp. 443-446. W. J. RUST. *Multiplication of Large Numbers.* Pp. 447-8. H. S. UHLER. *Construction of the Regular Heptagon by a Quartic Curve.* Pp. 449-450. C. B. HALDEMAN. *Problems discussed by Huygens.* Pp. 468-480. R. C. ARCHIBALD.

Jan. 1922.

Cuspidal Envelope Rosettes. Pp. 6-8. W. F. RIGGE. *Sexagesimal Fractions among the Babylonians.* Pp. 8-10. F. CAJORI. *A Budget of Exercises on Determinants.* Pp. 10-14. SIR T. MUIR. *Among my Autographs.* Pp. 14-16 [Sylvester: Lewis Carroll]. D. E. SMITH. *The Area of a Quadrilateral.* Pp. 29-36. R. C. ARCHIBALD.

Annals of Mathematics. (Princetown Univ., New Jersey, U.S.A.)

Sept. 1921.

On Matrices whose Elements are Integers. Pp. 1-15. O. VEBLEN and P. FRANKLIN. *An Algorism for Differential Invariant Theory.* Pp. 16-28. O. E. GLENN. *The General Theory of Cycle-harmonic Curves.* [The locus of the resultant motion of simple-harmonic motion in a straight line compounded with uniform rotatory motion about a fixed point in the line is a cycle-harmonic curve.] Pp. 29-39. R. E. MORITZ. *More Theorems on the Complete Quadrilateral.* Pp. 40-44. J. W. CLAWSON. *A Theorem on Cross-ratios in the Geometry of Inversion.* Pp. 45-51. J. L. WALSH. *The Condition for an Isothermal Family on a Surface.* Pp. 52-55. J. K. WHITTEMORE. *The Reversion of Class Number Relations and the Total Representation of Integers as Sums of Squares or Triangular Numbers.* Pp. 56-67. E. T. BELL. *Note on the term Maximal Subgroup.* Pp. 68-69. G. A. MILLER. *Reducible Cubic Forms expressible rationally as Determinants.* Pp. 70-74. L. E. DICKSON. *Note on the Picard Method of Successive Approximations.* Pp. 75-77. D. JACKSON. *A Fundamental System of Covariants of the Ternary Cubic Form.* Pp. 78-82. L. E. DICKSON. *The Modular Theory of Polyadic Numbers.* Pp. 83-90. A. A. BENNETT. *Some Analogies in Matric Theory.* Pp. 91-96. A. A. BENNETT. *Generalised Conjugate Matrices.* Pp. 97-100. P. FRANKLIN.

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